Revealing small-scale structures in turbulent Rayleigh-Bénard convection

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Thermal convection

- Free convection
  - imposed temperature gradient leads to density difference in a fluid
  - hot fluid tends to rise, cold fluid tends to fall
  - flow is driven by buoyancy

- Applications
  - kitchen:
    - boiling water in a kettle
    - air flow in an oven
  - atmosphere and ocean:
    - formation of cloud and thunderstorms
    - oceanic deep convection $\rightarrow$ moderate winter climate in northern Europe
  - Earth’s interior: mantle convection
Rayleigh-Bénard convection

Fluid in a box heated from below and cooled from above

- Rayleigh number
  \[ \text{Ra} = \frac{\alpha g H^3 \Delta T}{\nu \kappa} \]

- Prandtl number
  \[ \text{Pr} = \frac{\nu}{\kappa} \]

- Aspect ratio
  \[ \Gamma = \frac{D}{H} \]

\( \nu \): viscosity
\( \kappa \): thermal diffusivity
\( \alpha \): volume expansion coefficient
**Left**: $Ra = 6.8 \times 10^8, \ Pr = 596$ (dipropylene glycol), $\Gamma = 1$


**Right**: $Ra = 2.6 \times 10^9, \ Pr = 5.4$ (water), $\Gamma = 1$

Global and local properties

- **Large-scale (global) quantities**, e.g. total heat transfer across the system

- **Small-scale (local) quantities**
  - structure of velocity and temperature fields
  - effects of thermal plumes
  - Tool: structure functions, e.g.

\[
S_{u}^{(p)}(r) = \langle |u(\vec{x} + \vec{r}) - u(\vec{x})|^p \rangle_{\vec{x}}
\]

\[
S_{T}^{(p)}(r) = \langle |T(\vec{x} + \vec{r}) - T(\vec{x})|^p \rangle_{\vec{x}}
\]

- expect different behaviour in the bulk and near the boundaries
Temperature structure functions

\[ S_T^{(p)}(r) = \langle |T(\vec{x} + \vec{r}) - T(\vec{x})|^p \rangle_{\vec{x}} \]

- probing activities at scale \( r \)
- larger \( p \) emphasizes more extreme events
- motivations from Kolmogorov-type phenomenology
- scaling behavior:
  \[ S_T^{(p)}(r) \sim r^{\zeta_T} \]

Given a time-series of measurement \( T(t) \) at a fixed location, one can define a time domain structure function:

\[ S_T^{(p)}(\tau) = \langle |T(t + \tau) - T(t)|^p \rangle_t \]

- Taylor’s frozen flow hypothesis \( \Rightarrow S_T^{(p)}(\tau) \sim \tau^{\zeta_T} \)
Cascade picture: passive scalar

\[ \begin{align*} 2r & \rightarrow r \rightarrow \frac{r}{2} \rightarrow \cdots \rightarrow \frac{r}{2^N} \rightarrow \varepsilon \\
\Pi(2r) & \rightarrow \Pi(r) \rightarrow \Pi\left(\frac{r}{2}\right) \rightarrow \cdots \rightarrow \frac{r}{2^N} \rightarrow \varepsilon \end{align*} \]

Energy and temperature variance transferred from large scales to small scales, eventually being dissipated at the smallest scales

\[ \varepsilon = \text{mean energy dissipation rate} \]

\[ \chi = \text{mean thermal dissipation rate} \]

- **no buoyancy**, energy transfer rate \( \Pi \) is scale independent

\[ \Pi = \varepsilon \quad \text{in the inertial range} \]

- relevant parameters are: \( \varepsilon, \chi, r \)

- Obukhov-Corrsin scaling:

\[ S_T^{(p)}(r) \sim \varepsilon^{-p/6} \chi^{p/2} r^{p/3} \]
Cascade picture: active scalar

\[ 2r \rightarrow r \rightarrow r/2 \rightarrow r/2^N \]

\[ \alpha g u_r T_r \]

\[
\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \alpha g T \hat{z}
\]

\[
\partial_t T + (\vec{u} \cdot \nabla) T = \kappa \nabla^2 T
\]

- **buoyancy is important**, \( \Pi(2r) \) is negligible at \( r \)

\[
\Pi(r) = \alpha g u_r T_r \quad \text{in the inertial range}
\]

- relevant parameters are: \( \alpha g, \chi, r \)

- Bolgiano-Obukhov scaling:

\[
S_T^{(p)}(r) \sim (\alpha g)^{-p/5} \chi^{2p/5} r^{p/5}
\]
Intermittency correction

- $\epsilon$ and $\chi$ varies significantly in space

- **Refined similarity hypothesis**: replace $\epsilon$ and $\chi$ by their local average over a ball of radius $r$ about $\vec{x}$, $\mathcal{B}(\vec{x}, r)$

  \[
  \epsilon_r(\vec{x}) = \langle \epsilon(\vec{x}') \rangle_{\vec{x}' \in \mathcal{B}}
  \]

  \[
  \chi_r(\vec{x}) = \langle \chi(\vec{x}') \rangle_{\vec{x}' \in \mathcal{B}}
  \]

- The scaling predictions become

  **OC (passive)**: $S_T^{(p)}(r) \sim \langle \epsilon_r^{-p/6} \rangle_{\vec{x}} \langle \chi_r^{p/2} \rangle_{\vec{x}} r^{p/3}$

  **BO (active)**: $S_T^{(p)}(r) \sim (\alpha g)^{-p/5} \langle \chi_r^{2p/5} \rangle_{\vec{x}} r^{p/5}$

- $\langle \epsilon_r^{-p/6} \rangle_{\vec{x}}$ and $\langle \chi_r^{p/2} \rangle_{\vec{x}}$ are $r$-dependent, hence modifying the scaling exponents of $S_T^{(p)}(r)$
Some previous experimental work

Early time-domain measurements
- Wu et al. (PRL 1990) reported BO scaling at the convection cell center (using helium gas)
- Niemela et al. (Nature 2000) found BO scaling at large $\tau$ and OC scaling at small $\tau$ (using similar Ra, Pr and $\Gamma = 0.5$ as in Wu et al. 1990)
- Skrbet et al. (PRE 2002) found no scaling range at all (using the same setup as Niemela et al. 2000 but with $\Gamma = 1$)
- Zhou & Xia et al. (PRL 2001) observed BO scaling at the cell center and an apparent OC scaling in the mixing zone (using water)

Recent space-domain measurements
- Sun et al. (PRL 2006) demonstrated that behaviour at the cell center does not obey BO scaling and is closer to OC scaling
- Kunnen et al. (PRE 2008) reported a possible BO scaling at larger scales

Difficulties in comparing experimental results to theory:
- limited scaling range
- validity of the frozen flow hypothesis
- anisotropy and inhomogeneity, ...
Conditional structure functions

Recall in the space-domain, \( \chi_r(\vec{x}) = \langle \chi(\vec{x}') \rangle_{\vec{x}' \in \mathcal{B}(\vec{x},r)} \)

- OC (passive): \( S_T^{(p)}(r) \sim \langle \chi_r^{p/2} \rangle_{\vec{x}} r^{p/3} \)
- BO (active): \( S_T^{(p)}(r) \sim \langle \chi_r^{2p/5} \rangle_{\vec{x}} r^{p/5} \)

In the time-domain, given the time-series \( T(t) \) and \( \chi(t) \)

Define: \( \chi_\tau(t) = \langle \chi(t') \rangle_{t' \in \mathcal{B}(t,\tau)} \)

- OC (passive): \( S_T^{(p)}(\tau) \sim \langle \chi_\tau^{p/2} \rangle_t \tau^{p/3} \)
- BO (active): \( S_T^{(p)}(\tau) \sim \langle \chi_\tau^{2p/5} \rangle_t \tau^{p/5} \)

Define the conditional structure functions:

\[ \hat{S}_T^{(p)}(\tau, X) = \langle |T(t + \tau) - T(t)|^p \mid \chi_\tau(t) = X \rangle_t \]

- OC (passive): \( \hat{S}_T^{(p)}(\tau, X) \sim X^{p/2} \tau^{p/3} \)
- BO (active): \( \hat{S}_T^{(p)}(\tau, X) \sim X^{2p/5} \tau^{p/5} \)
Measuring local thermal dissipation rate

\[ \chi_{\tau}(\vec{x}, t) = \frac{1}{\tau} \int_t^{t+\tau} \kappa |\nabla T_f(\vec{x}, t')|^2 \, dt' \]

where \( T_f = \) temperature fluctuation

Home-made temperature gradient probe
- four temperature sensors of diameter 0.11mm
- separation between sensors = 0.25mm
- temperature resolution \( \sim 5\text{mK} \)

He & Tong, Phys. Rev. E 79, 026306 (2009)
Results: conditional structure functions

\[ \hat{S}_T^{(p)}(\tau, X) = \langle |T(t + \tau) - T(t)|^p \mid \chi_\tau(t) = X \rangle_t \sim X^{\beta(p)} \]

We have found significant scaling ranges in both cases.

\[ Ra = 8.3 \times 10^9, \ Pr = 5.5, \ \Gamma = 1 \]
Results: the scaling exponents

\[ \hat{S}_T^{(p)}(\tau, X) = \langle |T(t + \tau) - T(t)|^p \rangle \left| \chi_\tau(t) = X \right \rangle_t \sim X^{\beta(p)} \]

\[ p = 0.5 \text{ to } 4 \text{ from bottom to top, } \tau_0 \text{ is the data sampling interval} \]

- \( \beta(p) \) depends on \( \tau \)
- for each \( p \), \( \beta(p) \) attains a maximum \( \beta_{\text{max}}(p) \)
Results: passive vs. active

Experimental data: cell center (circles)  
bottom plate (triangle)

Theory: \( p/2 \) passive OC scaling (solid)  
\( \frac{2p}{5} \) active BO scaling (dashed)
introduce the conditional structure functions

\[ \hat{S}_T^{(p)}(\tau, X) = \langle |T(t + \tau) - T(t)|^p \bigg| \chi_\tau(t) = X \rangle_t \]

\( \chi_\tau = \) local time-averaged thermal dissipation rate

investigate the scaling with \( X \) (rather than \( \tau \)) and found significant scaling ranges,

\[ \hat{S}_T^{(p)}(\tau, X) \sim X^{\beta(p)} \]

results using experimental data at \( Ra = 8.3 \times 10^9 \) suggest that temperature obeys the

- the Obukhov-Corrsin scaling for a passive scalar at the convection cell center
- the Bolgiano-Obukhov scaling for an active scalar near the bottom plate