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On the formation and maintenance of the stratospheric surf zone as inferred from the zonally averaged potential vorticity distribution

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The diagnostic relation between eddy potential vorticity flux and Eliassen–Palm flux convergence in the transformed Eulerian mean shallow-water system is used to infer the flux convergence needed to establish a stratospheric surf zone comprising a single region of perfectly homogenized potential vorticity, or to maintain the surf zone in steady state against the restoring effect of radiative relaxation. In the transient case, and when wave breaking is assumed to mix potential vorticity on time-scales shorter than the radiative time-scale, the required flux convergence is an order of magnitude larger than that required to maintain the flow in steady state away from radiative equilibrium.

Key Words: potential vorticity; surf zone; Eliassen–Palm flux

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1. Introduction

The zonal mean distribution of momentum and temperature in the middle atmosphere, as well as the strength and shape of the residual Brewer–Dobson circulation, is determined to a large extent by the balance of two competing effects, thermal heating and cooling on the one hand, and the flux of heat and momentum arising from large-scale wave breaking on the other. In the winter stratosphere, cooling in the polar night implies low radiative equilibrium temperatures and, through thermal wind balance, a strong cyclonic polar vortex. Eddy heat and momentum fluxes from planetary-scale Rossby waves act to keep polar temperatures higher, and tropical temperatures lower, than the equilibrium state. The resulting temperature structure is consistent with a secondary or residual circulation, consisting of rising motion in the Tropics, mean poleward motion in midlatitudes, and descending motion over the winter pole. In addition to eddy fluxes from breaking planetary waves, those from breaking gravity waves in the upper stratosphere and mesosphere and from breaking synoptic-scale waves on the upward extension of the subtropical jet contribute to the overall shape and strength of the circulation (e.g. Butchart *et al.*, 2006; Shepherd, 2008, give recent overviews).

The combined action of eddy heat and momentum flux convergences is most naturally expressed in the framework of the transformed Eulerian mean (Andrews and McIntyre, 1976), where their effect on the flow may be represented as a single zonal force (strictly speaking, a torque) $\mathcal{F} = \nabla \cdot \mathbf{F}$ acting on the zonal momentum equation, where $\mathbf{F}$ is the Eliassen–Palm (EP) flux (Andrews and McIntyre, 1976; Andrews *et al.*, 1987). Such a force is balanced both by a zonal acceleration $\overline{u}_t$ and by a residual circulation $(\overline{v}^*, \overline{w}^*)$ in the height–latitude plane. It may be thought of as the torque required to balance angular momentum transport associated with the persistent meridional flow of air across nearly vertical surfaces of constant angular momentum. When eddy mixing occurs primarily as the result of Rossby wave breaking, $\mathcal{F}$ may be shown to be equivalent to the meridional eddy flux of potential vorticty (e.g. Andrews *et al.*, 1987). In the steady state, or long-time average, lateral advection of background potential vorticity by the residual circulation thus acts to cancel exactly the eddy potential vorticity fluxes associated with the mixing.

The dynamical nature of potential vorticity is such that its mixing is typically inhomogeneous: mixing is favoured in regions of weak latitudinal gradients of zonal mean potential vorticity such as the surf zone, and inhibited in regions of strong gradients such as the polar vortex edge. This provides a positive feedback in which the limiting case is a staircase-like profile (McIntyre, 1982; Marcus, 1993; Dunkerton and Scott, 2008) comprising a monotonic, piecewise uniform distribution in latitude. In the winter stratosphere, such a distribution may be identified in the polar vortex edge, surf zone, and subtropical barrier. In this case, the polar vortex edge and subtropical barrier act as wave guides for Rossby waves that are evanescent in latitude. Because of the lack of scale separation between background state and waves, the concept of lateral wave propagation is to a certain extent ill-defined, although EP flux vectors still meaningfully describe a transfer of wave activity from polar vortex edge to surf zone or subtropical barrier. The surf zone exists as a nonlinear Rossby wave critical layer for the waves on the flanking edges (McIntyre and Palmer, 1984) and may act as a sink of wave activity that grows over time as the result of continued wave forcing and the entrainment of air with higher or lower potential vorticity from either side (Scott and Tissier, 2012). Althought the dominant wave motions are those on the high-latitude vortex edge, the effect of wave breaking occurs over a broad latitudinal region that under certain conditions may extend deep into the Tropics (Dunkerton and O'Sullivan, 1996).

While the surf zone formation may be considered naturally in terms of potential vorticity mixing, the question arises as to how the EP flux convergence, acting over a certain time interval, may give rise to a given potential vorticity profile. In particular, for a given zonal mean potential vorticity profile, we seek to find the flux convergence, and associated residual circulation, that would be required to generate this profile from a state of rest, or to maintain it against the restoring effect of diabatic cooling. Because of the diagnostic relation between eddy potential vorticity fluxes and EP flux convergence, this question may be answered relatively simply subject to certain assumptions concerning the time history of the mixing. Such an approach may be considered complementary to that of downward control (Haynes *et al.*, 1991), where $\mathcal{F} = \nabla \cdot \mathbf{F}$ is specified and the residual circulation is inferred under steady conditions. One motivation for this note is that the arbitrary specification of $\mathcal{F}$ may easily lead to a potential vorticity distribution that is non-monotonic in latitude, and which would be not be physically realizable as the result of typical mixing processes.

In this note, we illustrate the approach in the context of a simple single-layer model, in which potential vorticity is taken to be perfectly mixed over a surf zone of prescribed latitudinal extent. Two cases are considered: (i) the transient situation, in which the potential vorticity distribution is assumed to arise as a result of the EP flux convergence acting over a fixed time interval; and (ii) the steady state situation, in which the potential vorticity distribution is maintained against the restoring action of radiative relaxation by the EP flux convergence. The transient case may be viewed as a simple model for the mixing processes involved during a stratospheric sudden warming event, or resulting from a sudden increase in tropospheric wave forcing. It is shown that, in the transient case, and when wave breaking is assumed to mix potential vorticity on time-scales shorter than the radiative time-scale, the required flux convergence is an order of magnitude larger than that required to maintain the flow in steady state away from radiative equilibrium.

2. Physical model

The use of a single-layer model is motivated by the observation that both zonal mean and upward propagating wave motions in the stratosphere are typically deep compared to the density scale height H. Consideration of Figure 8 of Matthewman *et al.* (2009) suggests a vertical phase tilt of the order of $\pi/3$ over the depth of the stratosphere for the case of wavenumber-1 stratospheric sudden warmings, while for the case of wavenumber-2 vortex splitting sudden warmings, the structure is close to the barotropic sudden warming considered in Esler and Scott (2005). In the latter case, the relevant vertical structure is the Lamb mode whose radius of deformation is equal to $NH/f_0\sqrt{\kappa(1-\kappa)}$, where N is the buoyancy frequency, f_0 is the Coriolis parameter, and $\kappa = 2/7$ for a diatomic gas (Matthewman and Esler, 2011). This is an extreme simplification, and details of the vertical structure of the stratospheric basic state and the vertical distribution of wave forcing affect many aspects of the stratospheric evolution. Nevertheless, we argue that the shallow-water model captures the leading order dynamical properties; in particular, it contains the main advective nonlinearity of the primitive equations together with the appropriate spherical geometry and balances of terms near the Equator. Secondary circulations are permitted through the effects of a mass source/sink term in the height equation, which may be considered as a simple representation of the effects of long-wave radiative cooling (Held and Phillips, 1990).

In the transformed Eulerian mean, the shallow-water equations take the form:

$$\frac{\partial \overline{u}}{\partial t} - \overline{v}^*(\overline{\zeta} + 2\Omega\mu) = \mathcal{F}, \tag{1a}$$

$$U\mu(U + 2\Omega a) = -g\,\frac{\partial \overline{h}}{\partial \mu}, \tag{1b}$$

$$\frac{\partial \overline{h}}{\partial t} + \frac{1}{a}\frac{\partial}{\partial \mu}\left\{\overline{h}\,\overline{v}^*(1-\mu^2)^{1/2}\right\} = -\alpha(\overline{h} - h_e), \tag{1c}$$

where a, g, and Ω are the Earth's radius, gravity and rotation rate, h is the layer depth, h_e is the layer depth at radiative equilibrium, α is the relaxation coefficient, $\mu = \sin\phi$ is sine of latitude, (u, v) are the zonal and latitudinal velocity components, $U = \overline{u}/\cos\phi$, and ζ is relative vorticity. Overbars denote the usual zonal mean, while stars denote the transformed Eulerian mean defined by $\overline{v}^* = \overline{hv}/\overline{h}$, equivalent to a mass-weighted zonal mean (Thuburn and Langneau, 1999).

If the departure from $\overline{v}^*$ is defined by $\hat{v} = v - \overline{v}^*$, then the zonal force on the right-hand side of Eq. (1a) may be shown to be equal to

$$\mathcal{F} = \overline{h\,\hat{v}\,\hat{Q}}, \tag{2}$$

the mass-weighted poleward eddy flux of potential vorticity, $Q = (2\Omega\mu + \zeta)/h$. The same term may be shown to be approximately equal to $\nabla \cdot \mathbf{F}/\overline{h}\cos\phi$, where $\nabla \cdot \mathbf{F}$ is the shallow-water version of the EP flux convergence (Thuburn and Langneau, 1999, give details, including a discussion of the closeness of the approximation in typical winter stratospheric conditions). The approximate relation is the shallow-water generalization of the Taylor identity (Taylor, 1915; Bretherton, 1966) relating EP flux convergence in the meridional plane to lateral eddy potential vorticity fluxes. It is exact under the quasi-geostrophic approximation, for which $\overline{v'q'} = \nabla \cdot \mathbf{F}$, where q is the quasi-geostrophic potential vorticity and primes denote departures from the zonal mean, and holds to a good approximation in the full primitive equations (e.g. Andrews *et al.*, 1987). The main point is that the distribution of $\mathcal{F}$ is here most naturally given in terms of the eddy mixing of potential vorticity, which in turn arises predominantly through the effects of breaking planetary waves in the surf zone.

In the absence of $\mathcal{F}$, the steady state solution to Eqs (1a)–(1c) is given by the radiative equilibrium height $\overline{h} = h_e$ and equilibrium velocity $\overline{u} = u_e$ related through gradient wind balance Eq. (1b). For simplicity, in the following we take $h_e = H$, the mean layer depth, so that the equilibrium solution is that of a resting atmosphere, $u_e = 0$. The mean layer depth is taken to be $H = 10$ km, giving a corresponding deformation radius comparable to that of the Lamb mode in the stratosphere.

3. The forcing distribution associated with a simple surf zone

3.1. *Mixing profile*

For a prescribed spatial and temporal distribution of $\mathcal{F}$, and relaxation rate α, the system Eqs (1a)–(1c) may in principle be solved to give the unique flow response $\overline{u}$, $\overline{h}$, and Q, toegether with the residual circulation $\overline{v}^*$. The solution is such that $\mathcal{F}$ is balanced in part by a local zonal acceleration $\overline{u}_t$, and in part by non-zonal advection of potential vorticity by $\overline{v}^*$. In the converse problem of interest here, only the instantaneous Q distribution is specified. The particular class of distributions considered here are similar to the staircase solutions of Dunkerton and Scott (2008), in which potential vorticity is perfectly mixed across a specified latitudinal band $[\mu_0, \mu_1]$, while outside this band it retains the value of the equilibrium planetary vorticity, thus taking the form

$$Q = \begin{cases} \Omega(\mu_0 + \mu_1) & \text{for } \mu_0 \le \mu \le \mu_1, \\ 2\Omega\mu & \text{otherwise,} \end{cases} \tag{3}$$

where μ_0 and μ_1 represent the southward and northward limits of the mixing zone, respectively. The aim is to solve Eqs (1a)–(1c) to find the corresponding $\mathcal{F}$ required either to generate this potential vorticity profile over a finite time, starting from a state of rest, or to maintain it in steady state against the action of radiative relaxation.

The first step is to invert the potential vorticity distribution to find the associated $\overline{u}$ and $\overline{h}$ profiles that are consistent with gradient wind balance Eq. (1b). This is done here using a simple iterative scheme, details of which are given in the Appendix. The profiles Q, $\overline{u}$ and $\overline{h}$ for representative values of μ_0 and μ_1 are shown in Figure 1. Here we have taken a fixed value of $\mu_1 = 0.9$, corresponding to a polar vortex edge at approximately $\phi = 62°$. Values of μ_0 range from $\mu_0 = 0.1$ to $\mu_0 = 0.5$, broadly consistent with the range of locations of the subtropical barrier in various winter conditions (e.g. Dunkerton and O'Sullivan, 1996). In all cases, the zonal flow $\overline{u}$ exhibits an approximately parabolic shape (which is exact in the barotropic case) with maximum negative velocity close to the centre of the mixed zone. Note that adding a polar vortex would substantially reduce the negative winds throughout much of the surf zone; the vortex has been omitted for clarity, however, since it has very little effect on the results of interest here.

3.2. *The transient problem*

In the transient problem we consider the mixed state Eq. (3) to have arisen at a given time T from a resting state $Q = 2\Omega\mu$ at $t = 0$, and seek the distribution of $\mathcal{F}$, and hence $\overline{v}^*$, which, acting over the interval $[0, T]$, may give rise to such a profile. It is clear that without some further assumption on the time evolution of $\mathcal{F}$, or the time evolution of the zonal mean flow response, the problem as specified is underdetermined: a number of possible choices of $\mathcal{F}(\mu, t)$ may give rise to the same final profile $Q(\mu, T)$. To obtain a plausible estimate of the time-averaged $\mathcal{F}(\mu)$ required to set up the final profile, we assume that the evolution is such that the width of the mixed zone increases linearly in time over the interval $[0, T]$, while remaining centred on the same mean latitude $\mu_c = (\mu_0 + \mu_1)/2$; that is, the mixed zone is bounded by latitudes

$$\begin{aligned} \mu_0'(t) &= \mu_c - (\mu_c - \mu_0)t/T\,, \\ \mu_1'(t) &= \mu_c + (\mu_1 - \mu_c)t/T\,. \end{aligned} \tag{4}$$

At each time, the potential vorticity profile is inverted as above to give the instantaneous velocity and height fields, from which their time derivatives are also obtained. Equation (1c) is then integrated in μ to give $\overline{v}^*$ from which Eq. (1a) gives $\mathcal{F}$, again at each instant t. The exact form of $\mathcal{F}$ depends on the assumptions made about the growth of the mixed region. However, here we are interested in the overall magnitude of the required $\mathcal{F}$ rather than in the detailed form. We note that an alternative assumption to Eq. (4) was tested, in which the fields $\overline{u}$ and $\overline{h}$, rather than the mixed zone, are assumed to grow linearly in time to their final profiles. The solution (not shown) is remarkably similar to that shown below, indicating that the form of the time-integrated $\mathcal{F}$ is insensitive to the precise details of the evolution of the mixed region.

Examples of the time-integrated $\mathcal{F}$ and $\overline{v}^*$ obtained from the mixing profiles of Figure 1(a), assuming the linear growth of Eq. (4), are reproduced in Figure 2. We have taken $T = 5$ days as a suitable time-scale corresponding approximately to the time-scale of tracer mixing following a stratospheric sudden warming, and have taken the radiative time-scale $\alpha^{-1} = 20$ days. For this value of T the dominant balance in Eq. (1a) is between $\overline{u}_t$ and $\mathcal{F}$: the form of $\mathcal{F}$ mostly follows that of $\overline{u}_t$ with relatively small corrections associated with $\overline{v}^*$. Note again that we are less concerned here with the precise shape of these profiles than with the overall magnitude of the response, which will be contrasted with that of the steady state solution next. In particular, we note that doubling the surf zone width, i.e. moving μ_0 from 0.5 to 0.1, results in an approximate quadrupling of $\mathcal{F}$, a dependence that is consistent with the Taylor identity Eq. (2): a doubling of the surf zone width implies a four-fold increase in the area between the step and the dotted line in Figure 1(a), and corresponding increase in the potential vorticity flux across the mid-point necessary to achieve the step (see also Figure 4 below). Note also that the dependence of $\overline{v}^*$ on μ_0 is even stronger, which may be attributed to the latitudinal dependence of the Coriolis parameter together with the extension of the wider surf zone into lower latitudes.

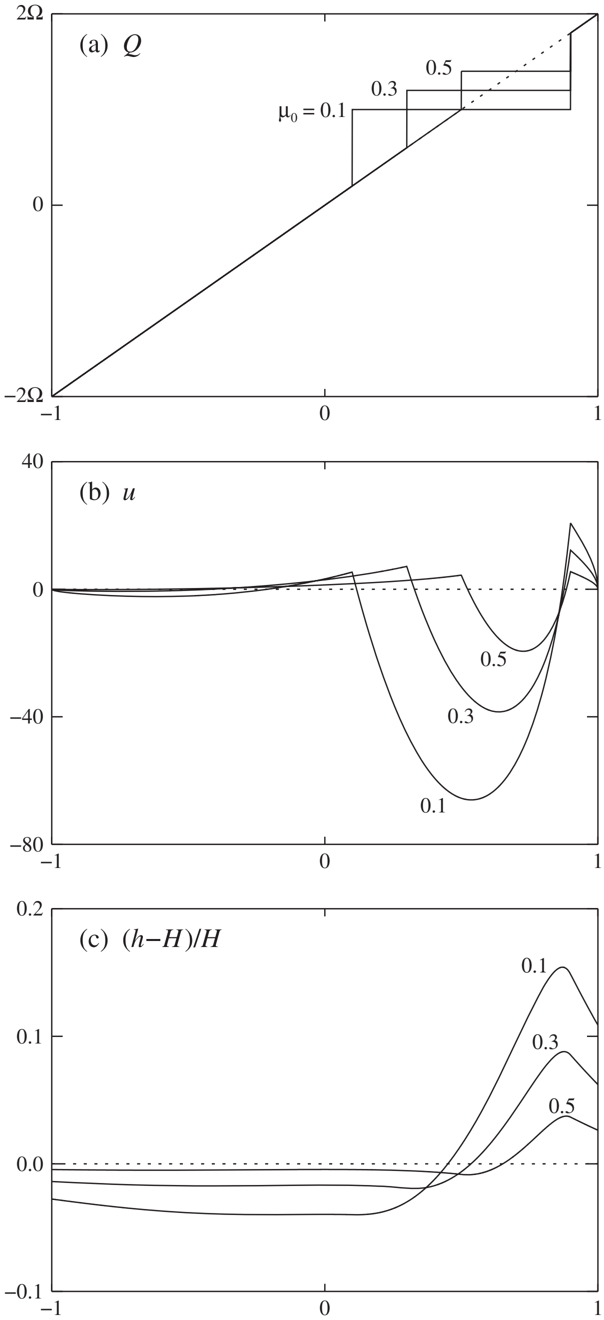

Figure 1. Simple surf zone consisting of a single mixed zone between latitudes μ_0 and μ_1, for $\mu_0 = 0.1, 0.3, 0.5$ and $\mu_1 = 0.9$, as given by Eq. (3): (a) potential vorticity as a function of μ, (b) associated zonal velocity $\overline{u}$ in units of m s^{-1}, and (c) associated perturbation height field.

3.3. *Steady state solutions*

Under steady conditions, the system Eqs (1a)–(1c) may be solved exactly for a given Q to obtain the $\mathcal{F}$ required to compensate the relaxation to the radiative equilibrium state. The time derivatives

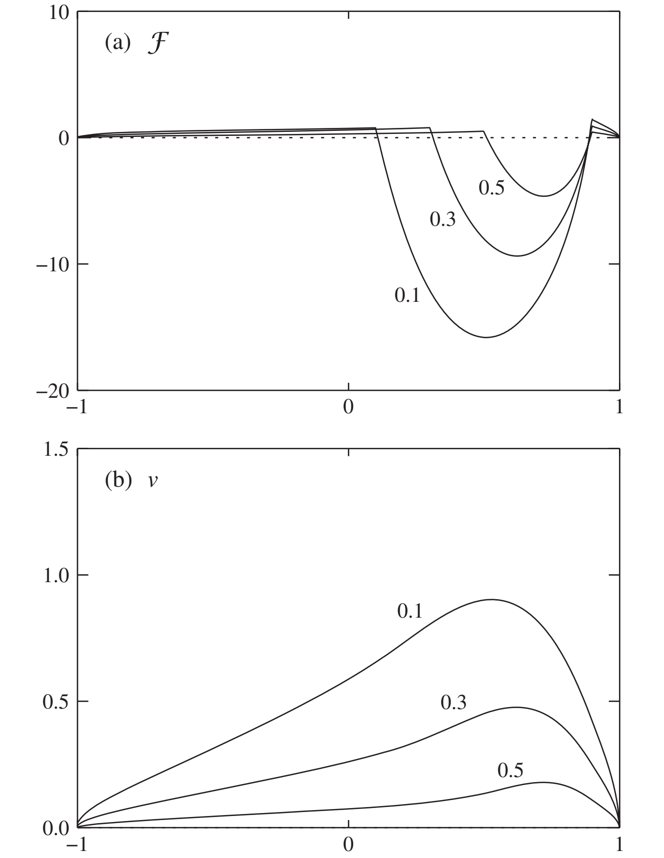

Figure 2. Solution to the transient problem with $T = 5$ days and $\alpha^{-1} = 20$ days: (a) average $\mathcal{F}$, in units of m s^{-1} day^{-1}, acting over the time interval $[0, T]$, required to produce the flow shown in Figure 1 while assuming a linear growth of the mixed zone, and (b) associated residual velocity $\overline{v}^*$ in units of m s^{-1}.

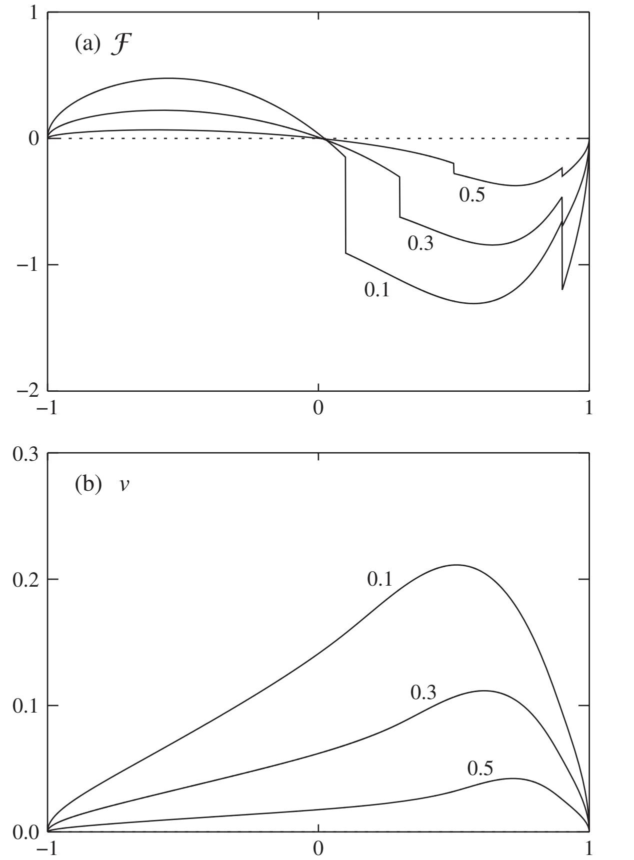

Figure 3. Solution to the steady problem with $\alpha^{-1} = 20$ days: (a) $\mathcal{F}$, in units of m s^{-1}day^{-1} required to maintain the flow shown in Figure 1, and (b) associated residual velocity $\overline{v}^*$ in units of m s^{-1}.

in Eqs (1a) and (1c) are dropped, so that $\overline{v}^*$ is obtained from the integral with respect to μ of the relaxation term, and $\mathcal{F}$ corresponds to the eddy potential vorticity flux required to balance the latitudinal advection of zonal mean potential vorticity. In other words, there is an exact balance between the terms $-\overline{v}^*(\overline{\zeta} + 2\Omega\mu)$ and $\mathcal{F}$ in Eq. (1a). Profiles thus obtained of $\mathcal{F}$ and $\overline{v}^*$ corresponding to the simple surf zone are presented in Figure 3. The main points worth emphasizing are:

- The magnitudes of both $\mathcal{F}$ and $\overline{v}^*$ obtained in the steady case are about an order of magnitude smaller than those obtained in the transient case.
- The maximum value attained by $|\mathcal{F}|$ in the range $[\mu_0, \mu_1]$ increases approximately quadratically with μ_0 for midlatitude values, as for the transient case above, with a slower increase as μ_0 approaches the Equator. The dependence on μ_0 is summarized in Figure 4; it is not particularly sensitive to the precise location, μ_1, of the northern boundary of the surf zone. Note that, although the curves are higher at higher μ_1, the values of max $|\mathcal{F}|$ for a given latitudinal extent of surf zone are not increasing with μ_1. For each value of μ_1 considered, the diamonds indicate the values of μ_0 and max $|\mathcal{F}|$ associated with a surf zone width of 30°.
- In the steady state, the departure of the height field from equilibrium h_e is directly proportional to the mass source or sink in the continuity equation (1c) and hence to the notional vertical velocity $\overline{w}^*$.
- For the cases $\mu_0 = 0.3$ and $\mu_0 = 0.1$, in both of which the surf zone penetrates well into the Tropics, substantial values of $\mathcal{F}$ are located across the Tropics and into the 'summer' hemisphere. At first sight this non-local $\mathcal{F}$ may appear inconsistent with the compact form of the surf zone, to which notional potential vorticity mixing is restricted: eddy potential vorticity fluxes appear to be required globally to maintain the local mixed zone between μ_0 and μ_1. The effect is a consequence of the non-local nature of the potential vorticity inversion and the simplified structure of the mixed zone: as was seen in Figure 1(c), the compact mixing zone is associated with a balanced height field that departs from the equilibrium height field across the whole sphere, implying a global $\overline{v}^*$. This circulation is necessarily associated with meridional advection of potential vorticity and so must be balanced by eddy mixing even outside the mixed zone. The profile of $\mathcal{F}$ may be made compact by the addition of a secondary mixed zone on the equatorward flank of the main surf zone.

4. Discussion

While the instantaneous potential vorticity distribution does not in itself provide information about eddy potential vorticity fluxes, the simple approach outlined above may be used to infer these fluxes given suitable assumptions about the time history of the flow. In particular, this approach may be used to infer the force $\mathcal{F}$ that is required to either establish a potential vorticity distribution corresponding to a simple mixed zone, or to maintain it against the restoring effects of radiative relaxation. In both the transient and steady state problems, there is a unique relation between the maximum value that $|\mathcal{F}|$ attains in the interval $[\mu_0, \mu_1]$ and the latitudinal extent of the surf zone. In the simple setting of the transformed Eulerian mean shallow-water system used here, the departure of the height field from equilibrium, $h - h_e$, may be associated with the vertical component of residual velocity, $\overline{w}^*$. For the case when the mixing region extends into the subtropics, $\overline{w}^*$ is found to have a local maximum on the Equator and penetrate non-locally into the summer hemisphere, a simple consequence of the non-local nature of the potential vorticity inversion in the shallow-water system. Perhaps more importantly, the zonal force $\mathcal{F}$ required to set up the mixed zone,

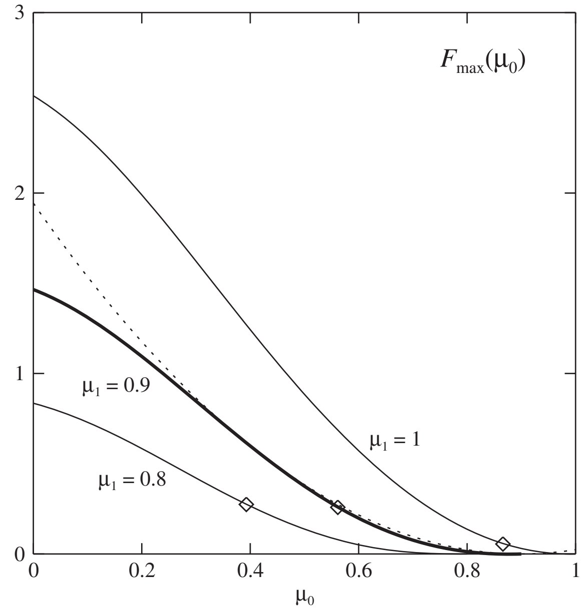

Figure 4. Dependence of the maximum value of $|\mathcal{F}|$ on μ_0 for values of $\mu_1 = 0.8$, 0.9 (bold), and 1; units are m s^{-1}day^{-1}. For reference only, the dotted line indicates the parabola $2.4(\mu_0 - \mu_1)^2$, considered as a function of μ_0, for $\mu_1 = 0.9$. Diamonds indicate, for each μ_1, the values of μ_0 and max $|\mathcal{F}|$ associated with a surf zone of latitudinal extent of 30°.

and the associated secondary circulation, is an order of magnitude larger than that required to maintain it in steady state. The results suggest that a well-defined surf zone may either develop as the result of persistent mixing over a long period, or as the result of a short episode of much stronger flux convergence, as might occur during a stratospheric sudden warming. Further, once formed, a surf zone may be maintained with relatively weak wave forcing.

While the shallow-water model is a severe simplification of stratospheric motions, it captures some of the main dynamical features in view, in particular, of the deep vertical structure of the dominant planetary-scale Rossby waves and the deep vertical structure of the stratospheric sudden warming. At the same time, it has been pointed out that details of the vertical distribution of wave forcing may influence such things as vortex recovery times following a sudden warming (Gerber *et al.*, 2009; Hitchcock *et al.*, 2013). In principle, the approach outlined above may be extended to the full 3D primitive equations: the EP flux divergence is again diagnostically related to the meridional flux of potential vorticity, although the inversion of the potential vorticity becomes more complicated and appropriate boundary conditions must be taken into account. Requiring a latitudinally monotonic potential vorticity distribution again places constraints on the wave forcing. Similar ideas were applied recently by Cohen *et al.* (2013) to the issue of mean flow forcing by gravity wave drag parametrizations in general circulation models, where such forcings are typically applied without consideration of the zonal mean potential vorticity profile.

We close with a few words relating the above considerations to the issue of planetary wave driving of the Brewer–Dobson circulation. The downward control principle of Haynes *et al.* (1991) causes some apparent difficulty in reconciling the observed distribution of the Brewer–Dobson circulation, in particular its tropical maximum, with the idea that the dominant wave forcing in the stratosphere is associated with planetary waves propagating on the edge of the polar vortex, which may be regarded as a high-latitude wave guide (Simmons, 1974; McIntyre, 1982; Esler and Scott, 2005). That much of the tropical $\overline{w}^*$ is related to high-latitude planetary waves, rather than to synoptic-scale Rossby waves or gravity waves has been suggested on the basis of both the observed Northern Hemisphere winter maximum in tropical heating rates (Yulaeva *et al.*, 1994), and the correlation of the interannual variability in tropical heating rates with high-latitude $\nabla \cdot \mathbf{F}$ (Ueyama and Wallace, 2010). The difficulty has in part been resolved by idealized numerical studies that have considered how $\overline{w}^*$ depends on details of the extension of $\mathcal{F}$ into the subtropics; relatively weak subtropical wave forcing may often be sufficient to drive significant tropical upwelling (Plumb and Eluszkiewicz, 1999; Tung and Kinnersley, 2001; Scott, 2002).

We make two observations. First, that while in idealized studies a useful approach has often been to consider the response of the transformed Eulerian mean equations to a specified compact distribution of the force $\mathcal{F}$, such a distribution may, when acting over a given time interval, result in a zonal mean flow that is inconsistent with the above mixing hypothesis, in the sense that the potential vorticity becomes non-monotonic in latitude. It is simple to verify numerically that even modest magnitudes of $\mathcal{F}$ of the order of 1 m s^{-1}day^{-1}, when applied over a restricted zone in midlatitudes will typically result in a potential vorticity profile that is non-monotonic in latitude, even if great care is taken to ensure the smoothness of the distribution (a similar estimate for the 3D stratified case was obtained by Cohen *et al.*, 2013). In contrast, regarding $\mathcal{F}$ as resulting from eddy mixing of potential vorticity implies that its distribution should not be specified arbitrarily, but consistently with the mixing itself; the latter is expected to homogenize potential vorticity but not to 'over mix' into a non-monotonic profile (McIntyre, 1982, 2008; McIntyre and Palmer, 1984; Killworth and McIntyre, 1985; Wood and McIntyre, 2010). The approach outlined above may provide a better framework in which to examine the secondary circulation response to different distributions of wave forcing in idealized models, requiring any specified $\mathcal{F}$ to be consistent with potential vorticity mixing.

Second, we observe that the presense of low-latitude $\mathcal{F}$ need not be associated with horizontal wave propagation, or an 'upwards and equatorwards' wave propagation: as discussed above, the weakness of the potential vorticity gradients in the surf zone means that lateral wave propagation is poorly defined there. While the dominant planetary-scale Rossby wave motions are in this sense trapped on the edge of the winter stratospheric polar vortex, the nonlinear saturation and breaking of these waves occurs in a broad surf zone, or Rossby wave critical layer, which may extend to considerably lower latitudes (e.g. McIntyre, 1982). Considering the flux of potential vorticity occurring in such a wave breaking process, as above, implies a distribution of EP flux convergence which must also extend well away from the high-latitude polar vortex edge and deep into subtropical or tropical latitudes. In other words, waves on the polar vortex edge may contribute to EP flux convergence well into the subtropics. In this way, the influence of high-latitude waves on the secondary circulation may be considered as a highly non-local effect, resulting from the inhomogeneous nature of wave breaking and associated eddy flux convergence, and whose details are determined by the width of the surf zone. These considerations suggest that an important step towards understanding the strength and distribution of the Brewer–Dobson circulation, both in the atmosphere and in more complicated models, will involve an improved understanding of the main dynamical processes which determine the latitudinal extent of the surf zone.

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Appendix

For completeness we briefly describe the iteration scheme used to obtain $\overline{u}$ and $\overline{h}$ from $Q = (2\Omega\mu + \zeta)/h$ subject to Eq. (1b). The method is based on the following system, where overbars have been dropped to simplify notation:

$$\zeta_n = Q\,h_{n-1} - 2\Omega\,\mu\,, \tag{A1a}$$

$$h_n = g^{-1}\int^{\mu} U_n\,\mu'(U_n + 2\Omega\,a)\,\mathrm{d}\mu', \tag{A1b}$$

with U_n obtained from ζ_n, and with $h_0 = H$. The constant of integration in Eq. (A1b) is chosen to ensure the global mean of h_n is always equal to H. To ensure that the global mean of ζ_n is identically zero at each iteration, Eq. (A1a) is modified by replacing Q with $Q + Q_0$, where Q_0 is a (dynamically irrelevant) constant defined by

$$Q_0 = -\frac{1}{H}\int_{-1}^{1} Q\,h_{n-1}\,\mathrm{d}\mu\,. \tag{A2}$$

Finally, to ensure convergence of the iteration, a successive relaxation step is included whereby for each $n > 1$ the estimate for ζ_n obtained from Eq. (A1a) is replaced by $w\zeta_n + (1-w)\zeta_{n-1}$, where w is a relaxation constant determined empirically for optimal convergence. A value of $w = 0.3$ was found to be appropriate for the parameter values used here; a smaller value is required to obtain convergence at smaller H.

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