Elliptic instabilities in vortices with axial flow

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A vortex which is elliptically deformed by an external strain field is generically subjected to the so-called elliptic instability. This instability has been extensively studied in vortices without axial flow. The purpose of this work is to analyse the effect of an axial flow on its occurence and determine the elliptic stability characteristics of a classical model of vortex with jet, the so-called Batchelor vortex.

The elliptic instability is now recognized as an important phenomenon of vortex dynamics. It is believed to take place in various contexts ranging from three-dimensional transition in shear flows to vortex interactions and flows in elliptic containers. We refer to the recent review [1] for details and references. The generic aspects of the elliptic instability were first identified by Pierrehumbert [2] and Bayly [3] who considered the local stability properties of an elliptic flow. Before these local analyses, an instability which develops in strained vortices had been identified [4, 5]. The first global stability analysis of the elliptic instability had been performed and an instability mechanism in terms of normal mode resonance had been provided. Moore & Saffman [4] showed for an arbitrary strained vortex without axial flow that two neutral normal modes (Kelvin waves) of the underlying axisymmetric vortex are coupled by the strain field if their characteristics satisfy a condition of resonance. They also provided, by an asymptotic analysis in the limit of small strain field, a formal expression for the growth rate of the resonant modes. This theory has been applied to various vortices without axial flow [5, 6, 7, 8].

The effect of axial flow has been considered only recently for the Rankine vortex with a constant axial flow in its core [9]. It was shown that axial flow modifies the characteristics of the most unstable resonant modes. However, the Rankine vortex is a crude approximation for a realistic vortex. In particular, we now know that some of its normal modes disappear when the vortex profile is changed into a smoothly varying profile [10, 11]. The vortex we consider here is a classical model of vortex with axial flow. It is known to model correctly the structure of trailing vortices in the far-wake of airplanes [12].

In the aeronautical context, the elliptic instability is expected to intervene in the dynamics of the multiple vortices generated by aircraft wings. Each vortex is in the strain field of surrounding vortices, and therefore subjected to the elliptic instability. In configurations without axial flow, the elliptic instability has been experimentally observed in both counter-rotating vortices [13] and in co-rotating vortices [14]. It has been modelled using Moore & Saffman approach in [15]. They demonstrated that this approach based on a single strained vortex provides very good estimates for the elliptic instability characteristics in vortex pairs. In the present work, a similar comparison will be performed: The theory constructed for a single strained vortex will be validated by numerical results obtained for a pair of counter-rotating Batchelor vortices.

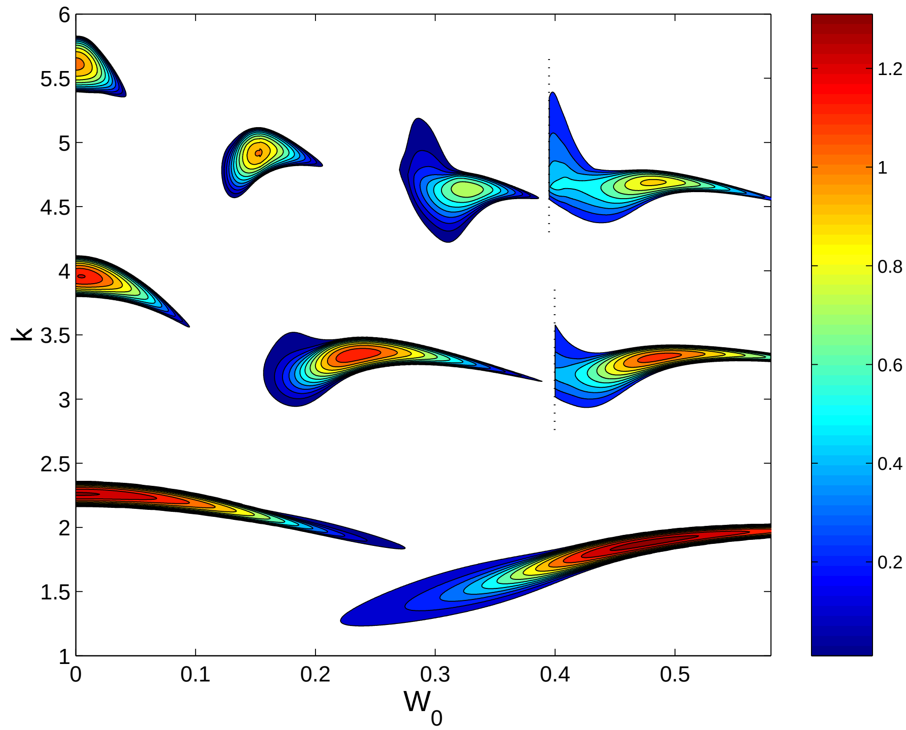

Figure 1: Instability area of the principal coupling modes in a plane (W_0, k). Colors correspond to the intensity of the growth rate (from blue to red: from minimum to maximum) for $Re = 20\,000$ and $\varepsilon = 0.01$.

Batchelor vortex has been the subject of numerous works. It is known to be unstable with respect to inviscid perturbations when the axial flow exceeds a critical value [16]. Here we are not interested in this instability. The axial flow will be varied below this critical value. Recently, Fabre & Jacquin [11] have discovered that Batchelor vortex also exhibits unstable modes for small axial flow if the Reynolds number is sufficiently large. These modes are purely viscous and localized in the vortex center. Their growth rate is $\mathcal{O}(Re^{-1/3})$. These peculiar modes will not be considered in the present work. We shall consider the resonant coupling of inviscid normal modes only. For small axial flow, these normal modes are expected to be either neutral or damped by a critical layer singularity. The appearance of critical layers is a common feature of normal modes in vortices with continous vorticity profiles but very few information is available in the literature [10, 17]. An important part of the present work is to dertermine these modes for the Batchelor vortex as axial flow is varied.

In this work we obtain the following results. We show that axial flow modifies the characteristics of the elliptic instability. Without axial flow, the elliptic instability mode is formed of two stationary symmetric Kelvin modes $m = 1$ and $m = -1$. Axial flow breaks the symmetry between the $m = 1$ and $m = -1$ Kelvin modes such that the elliptic instability is no longer a sinuous stationary deformation in presence of a small axial flow. For larger axial

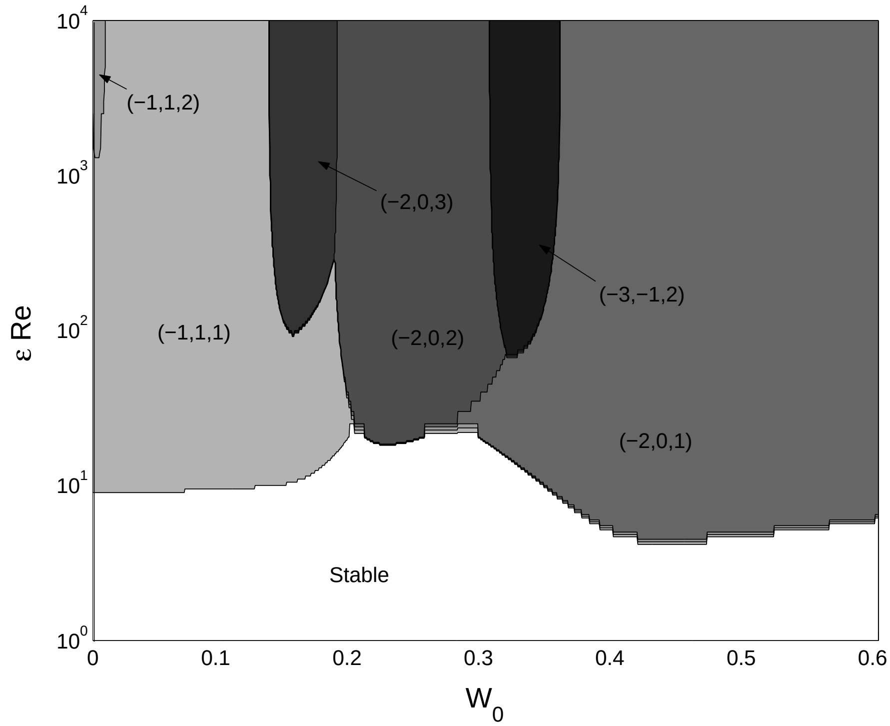

Figure 2: Most unstable principal mode in the $(W_0, \varepsilon Re)$ plane for $\varepsilon = 0.01$.

flow, the resonance between Kelvin modes $m = 1$ and $m = -1$ disappears because one of the two modes becomes strongly damped due to a critical layer singularity. However, another resonance between $m = 0$ and $m = -2$ becomes possible leading to a new instability mode. As the axial flow is progressively increased, this resonance is replaced by another between modes $m = -1$ and $m = -3$ and so on. Typically, for a fixed strain rate ε and a fixed Reynolds number, the theory predicts an instability diagram of the form shown in figure 1 where W_0 is the axial flow parameter and k the axial wavenumber of the perturbation. Each island corresponds to a different instability mode. The characteristics of the most unstable mode over all the wavenumber are shown for the same value of the strain rate ε in figure 2. The first two indexes indicate the azimuthal wavenumbers of the resonant Kelvin modes, the third index the branch number.

A comparison is also made with results obtained by direct numerical simulations of two counter-rotating Batchelor vortices. A very good agreement with the theory is demonstrated.

References

[1] R. R. Kerswell, "Elliptical instability," Annu. Rev. Fluid Mech. **34,** 83–113 (2002).

[2] R. T. Pierrehumbert, "Universal short-wave instability of two-dimensional eddies in an inviscid fluid," Phys. Rev. Lett. **57,** 2157–2160 (1986).

[3] B. J. Bayly, "Three-dimensional Instability of Elliptical Flow," Phys. Rev. Lett. **57,** 2160–2163 (1986).

[4] D. W. Moore and P. G. Saffman, "The instability of a straight vortex filament in a strain field," Proc. R. Soc. Lond. A. **346,** 413–425 (1975).

[5] C.-Y. Tsai and S. E. Widnall, "The stability of short waves on a straight vortex filament in a weak externally imposed strain field," J. Fluid Mech. **73,** 721–733 (1976).

[6] C. Eloy and S. Le Dizès, "Stability of the Rankine vortex in a multipolar strain field," Phys. Fluids **13,** 660–676 (2001).

[7] C. Eloy and S. Le Dizès, "Three-dimensional instability of Burgers and Lamb-Oseen vortices in a strain field," J. Fluid Mech. **378,** 145–166 (1999).

[8] D. Fabre and L. Jacquin, "Short-wave cooperative instabilities in representative aircraft vortices," Phys. Fluids **16,** 1366–1378 (2004).

[9] L. Lacaze, A.-L. Birbaud, and S. Le Dizès, "Elliptic instability in a Rankine vortex with axial flow," Phys. Fluids 17 (2005).

[10] D. Sipp and L. Jacquin, "Widnall instabilities in vortex pairs," Phys. Fluids **15,** 1861–1874 (2003).

[11] D. Fabre, D. Sipp, and L. Jacquin, "The Kelvin waves and the singular modes of the Lamb-Oseen vortex," J. Fluid Mech. (2004), (in press).

[12] G. K. Batchelor, "Axial flow in trailing line vortices," J. Fluid Mech. **20,** 645–658 (1964).

[13] T. Leweke and C. H. K. Williamson, "Cooperative elliptic instability of a vortex pair," J. Fluid Mech. **360,** 85–119 (1998).

[14] P. Meunier and T. Leweke, "Elliptic instability of a co-rotating vortex pair," J. Fluid Mech. (2005), (in press).

[15] S. Le Dizès and F. Laporte, "Theoretical predictions for the elliptic instability in a two-vortex flow," J. Fluid Mech. **471,** 169–201 (2002).

[16] R. L. Ash and M. R. Khorrami, "Vortex Stability," in *Fluid Vortices*, S. I. Green, ed., (Kluwer Academic Publishers, 1995), Chap. VIII, pp. 317–372.

[17] S. Le Dizès and L. Lacaze, "An asymptotic description of vortex Kelvin modes," J. Fluid Mech. (2005), in press.