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Quasi-geostrophic shallow-water vortex–patch equilibria and their stability

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We examine the equilibrium form, properties, stability and nonlinear evolution of steadily-rotating simply-connected vortex patches in the single-layer quasi-geostrophic model of geophysical fluid dynamics. This model, valid for rotating shallow-water flow in the limit of small Rossby and Froude numbers, has an intrinsic length scale $L_D$ called the “Rossby deformation length” relating the strength of the stratification to that of the background rotation. Here, we generate steadily-rotating vortex equilibria for a wide range of $\gamma = L/L_D$, where $L$ is the typical horizontal length scale of the vortex. We vary both $\gamma$ (over the range $0.02 \leq \gamma \leq 10$) and the vortex aspect ratio $\lambda$ (over the range $0 < \lambda < 1$). We find two modes of instability arising at sufficiently small aspect ratio $\lambda < \lambda_c(\gamma)$: an asymmetric (dominantly wave 3) mode at small $\gamma$ (or large $L_D$) and a symmetric (dominantly wave 4) mode at large $\gamma$ (or small $L_D$). At marginal stability, the asymmetric mode dominates for $\gamma \approx 3$, while the symmetric mode dominates for $\gamma > 3$. The nonlinear evolution of weakly-perturbed unstable equilibria results in major structural changes, in most cases producing two dominant vortex patches and thin, quasi-passive filaments. Overall, the nonlinear evolution can be classified into three principal types: (1) vacillations for a limited range of aspect ratios $\lambda$ when $5 \leq \gamma \leq 6$, (2) filamentation and a single-dominant vortex for $\gamma \approx 3$, and (3) vortex splitting – asymmetric for $1 \leq \gamma \leq 4$ and symmetric for $\gamma \gtrsim 4$.

Keywords: Quasi-geostrophic shallow-water; Vortex–patches; Equilibria

1. Introduction

Vortices or eddies are commonly-occurring highly-nonlinear dynamical features of the Earth’s oceans and atmosphere, and of the atmospheres of the planets. Ebbesmeyer et al. (1986) estimates that there are over 10,000 vortices in the surface layers of the North Atlantic alone. Examples of terrestrial atmospheric vortices include such long-lived features as mid-latitude cyclones and the “polar vortex”, which dominates the extratropical winter stratosphere (Norton 1994). Numerical simulations of weakly-forced and damped geophysical flows over the past three decades indicate that vortices emerge spontaneously from incoherent motions and subsequently dominate the “turbulent” flow evolution (cf. McWilliams 1984, and many others since), under
conditions when large-scale planetary vorticity gradients may be neglected. A special case of vortices are dumbbell-shaped ones, as shown in figures 1 and 2. These occur both in experimental studies (Meunier and Leweke 2001, Cerretelli and Williamson 2003b) and in simulations of turbulent flows. Figure 1 shows such a vortex which forms from an initially turbulent field in a numerical simulation and persists for many rotation periods. Additionally, dumbbell-shaped structures may also be found in the Earth’s atmosphere. For example, during the “sudden stratospheric warming” phenomenon, a dumbbell structure appears and lasts a few days before breaking up (Rosier et al. 1994). Furthermore, given the variety of different structures found in the ocean, it is also likely that oceanic dumbbell-shaped vortices exist.

Vortices occur over a vast range of spatial and temporal scales, and their interactions can be exceedingly complex (cf. Dritschel and Scott 2009). To be able to better understand their fundamental properties, idealisations need to be made. Here, following many previous studies, we focus on one of the simplest geophysical fluid dynamical models, namely the quasi-geostrophic shallow water model (Polvani et al. 1989, Waugh

Figure 1. An example of the formation of dumbbell-shaped vortices in two-dimensional quasi-geostrophic turbulence (simulated using the CLAM method (Dritschel and Fontane 2010) at an effective resolution of 4096\textsuperscript{2}). Here, the domain has dimensions 2\pi by 2\pi, and we have the Rossby deformation wavenumber \( k_D = 1/L_D = 10 \) and \( L \approx 0.071 \) (calculated from the area of the dumbbell-shaped vortex in the right-hand panel), giving \( L/L_D \approx 7 \). Starting from a random-phased PV field peaked at wavenumber \( k_0 = 20 \) (shown on the left at \( T = 100 \) in units of \( T = L_D/u_{\text{rms}}(0) \)), we let the field decay freely for 20000 time units. Halfway through the simulation, at roughly \( T = 10750 \), a dumbbell-shaped state forms, which persists until the end of simulation (for almost 10000 time units). This is shown at time \( T = 15000 \) on the right.

Figure 2. Schematic diagram of a vortex patch equilibrium.
and Dritschel 1991). The key feature of this model is the Rossby deformation length $L_D$, embodying the effects of rotation and stratification in a succinct way. The importance of the Rossby deformation radius, defined to be $L_D = c/f$ (where $c$ is the short-scale gravity wave speed, and $f$ is the Coriolis frequency), is that fluid motions at scales $L \ll L_D$ behave in the classical two-dimensional (2D) manner, with negligible free-surface deformations (as in the 2D Euler equations), while on the other hand, motions at scales $L \gg L_D$ are strongly affected by free-surface deformations, and become confined to fronts or jets of width $O(L_D)$. In the oceans, typical $L_D$ values range from 25 to 100 km, values comparable to the radius of many ocean eddies such as Gulf Stream “rings”, “meddies” or “swoddies” (Carton 2001). In the Earth’s atmosphere at mid-latitudes and poleward, typical $L_D$ values are an order of magnitude larger, ranging from 1000 to 1500 km (Charney and Flierl 1981, Juckes and McIntyre 1987). In Jupiter’s atmosphere, indirect modelling estimates suggest $L_D$ is 1/40th of the planet’s radius (Scott and Polvani 2008).

In this article, we investigate how the ratio $\gamma = L/L_D$ affects the dynamics of a flow in an especially simple context, namely in the equilibrium shape and stability of an isolated, two-fold symmetric patch of uniform PV. The barotropic case $\gamma = 0$ is well understood; in this case, the flow is described by the 2D Euler equations, and the equilibria take the form of ellipses, $x^2/a^2 + y^2/b^2 = 1$ (Kirchhoff 1876). These (relative) equilibria rotate at a constant rate $\Omega$ which depends only on the aspect ratio $\lambda = b/a$ and the uniform PV $q_0$ (here simply vorticity): $\Omega = q_0 \lambda / (1 + \lambda)^2$. Love (1893) showed that these equilibria are linearly unstable if $\lambda < 1/3$, with the first mode of instability having an asymmetric wave-3 form (in elliptic coordinates). This instability has since been confirmed in the fully nonlinear equations (see, e.g. Dritschel 1986), and extensive studies examining the evolution of unstable elliptical vortices have been done (see, e.g. Polvani and Flierl 1986, Mitchell and Rossi 2008 and others).

Much less is known when $\gamma > 0$. Polvani et al. (1989) developed a numerical procedure to compute two-fold symmetric equilibria for a selected set of parameters, specifically $\lambda = 0.6, 0.4$ and 0.286, each for 21 $\gamma$ values equally spaced on a logarithmic scale between $10^{-2}$ and $10^{2}$. They found that the finite-$L_D$ ($\gamma > 0$) equilibria are qualitatively different from ellipses, with these differences becoming more pronounced at smaller aspect ratios $\lambda$. Our purpose here is to extend their work first by a more comprehensive coverage of parameter space, and then by studying the linear and nonlinear stability of the equilibria (it turns out that most instabilities occur for $\lambda < 0.286$). High-resolution nonlinear simulations permit us to examine the fate of instabilities, and to provide a deeper understanding of the effects of finite $L_D$.

This article is organised as follows. We first review the physical system, present the numerical method used to find the equilibria, and then discuss the shapes and properties of the equilibria in section 2. This is followed by a linear stability analysis in section 3 and by nonlinear numerical simulations in section 4. We conclude in section 5 with a summary of the main findings and some ideas for further study.

2. The flow model and vortex–patch equilibria

2.1. Quasi-geostrophic flow

The quasi-geostrophic shallow-water (QGSW) model is perhaps the most popular model to date for the study of fundamental aspects of atmospheric and oceanic
(geophysical) flows (Vallis 2008, p. 207). This is due to the model’s great simplicity: two-dimensional, no gravity waves (“balanced”), versatility. Yet, the QGSW model embodies key elements of geophysical flows: potential-vorticity, vortices, fronts, jets and turbulence. The QGSW model is, despite its simplicity, nonlinear \textit{and} parameter rich. Here, we examine this model in its simplest form: no forcing, no damping, no topography and constant planetary vorticity $f$. The corresponding QGSW model consists of a single “prognostic” equation for the material (conservative) advection of quasi-geostrophic potential vorticity “QGPV” $q$,

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} + v \frac{\partial q}{\partial y} = 0,$$

(1)

and a Helmholtz-type “inversion relation” providing the (non-divergent) flow field $u = (u, v)$ from $q$,

$$(\nabla^2 - L_D^{-2})\psi = q, \quad u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x},$$

(2)

where $L_D = c/f$ is the Rossby deformation length.

2.2. The vortex patch model

We consider the simplest form of a vortex, namely a patch of uniform (QG)PV, $q = q_0$, in an unbounded domain with $q = 0$ outside the vortex. The vortex is entirely prescribed by its boundary shape, and we seek the shapes which are preserved under the dynamical evolution, i.e. steadily rotate. (Note, these are often referred to as “relative equilibria”, but here we will simply call them “equilibria”.) Many possible shapes are likely to exist, depending on the symmetry imposed (or not). Here, we seek two-fold symmetric simply-connected shapes, analogous to the elliptical vortices found in the barotropic limit $\gamma = L/L_D = 0$. An example is given in figure 2.

The simplicity of a vortex patch carries over to the calculation of its induced velocity field. As shown by Deem and Zabusky (1978a,b), the dynamics of a vortex patch depends only on the shape of its bounding contour, a property coined “contour dynamics”. Using Stokes’ theorem, one can reduce the double integral involved in inverting the operator $\nabla^2 - L_D^{-2}$ to a single contour integral, resulting in

$$\frac{dx}{dt} = u(x) = \frac{q_0}{2\pi} \int_{C} K_0(|x - x'|/L_D)dx',$$

(3)

where $K_0$ is the modified Bessel function of zeroth order, $C$ is the bounding contour, drawn in a right-handed sense, and $x' \in C$. When $x \in C$ also, this equation completely describes the motion of $C$. It may be generalised to any number of contours $C$ by simple linear superposition (Dritschel 1989).

The equilibria are generated following the iterative numerical procedure discussed in Dritschel (1995). Without loss of generality, the PV inside the vortex is set to $q_0 = 1$, and the area of the vortex is set to $A = \pi$ (the latter implies that the characteristic length scale $L = 1$). Then, starting from a circular patch, which is a known equilibrium for any value of $\gamma = L/L_D$, we slightly decrease the aspect ratio $\lambda$ by $\Delta\lambda = 0.001$, and find a new equilibrium, together with its rotation rate $\Omega$, by iteration until convergence, here when the maximum normal variation of the contour shape is less than $10^{-9}$. We then reduce $\lambda$.
further and repeat the procedure until the method no longer converges (this normally occurs for very small \( \lambda \) between 0.003 and 0.007, depending on \( \gamma \)). This generates a family of equilibria varying with \( \lambda \) for a fixed value of \( \gamma \). To start the procedure, we need a guess for the rotation rate \( \Omega \) relevant to a near circular vortex \( (1 - \lambda \ll 1) \). From a linear stability analysis (appendix A), it can be shown that for an \( m \)-fold symmetric wave \( \Omega = I_1(\gamma)K_1(\gamma) - I_m(\gamma)K_m(\gamma) \), where \( I_m \) and \( K_m \) are the modified Bessel functions of the \( m \)th order. Note that we confine our attention to \( m = 2 \).

Typically, 400 nodes are used to represent the vortex boundary \( C \). These are connected together by local cubic splines to achieve high accuracy, as discussed in Dritschel (1988). 800 nodes are used when \( \gamma > 1 \), in part to accurately capture the weak instability occurring for very small \( \lambda \) (when the vortex is strongly distorted), and in part for long-time accuracy in the nonlinear simulations (the evolution slows down markedly as \( \gamma \) increases beyond 1, see below). The difference in the vortex shape between 400 and 800 nodes, however, is much smaller than the line width plotted in figure 2.

We generate families of equilibria for \( \gamma = 0.02, 0.25, 0.5 \) and thereafter in increments of \( \Delta \gamma = 0.5 \) to \( \gamma = 10 \) (22 families in total). This range more than sufficiently encompasses the range of values thought to characterise vortices in the oceans, the atmosphere and in other planetary atmospheres. Note that the barotropic case corresponds to \( \gamma = 0 \).

### 2.3. Properties of the equilibria

A few examples of the equilibrium contour shapes are presented in figure 3 for three different values of \( \gamma \). In each frame, we illustrate three different aspect ratios: \( \lambda = 0.5 \), the aspect ratio at marginal stability \( \lambda_c \) (section 3), and the smallest aspect ratio for which we achieve convergence \( \lambda_f \). Here, and for all \( \gamma \) investigated, the shape deforms into a dumbbell shape, nearly pinching off as \( \lambda \to 0 \). It is likely that the limiting form for \( \lambda = 0 \) is a pair of vortices touching at the origin (see Polvani et al. (1989) for comparable examples of doubly-connected equilibria). Note that as \( \gamma \) increases, the equilibria become less elongated in \( x \) with decreasing \( \lambda \). This is due to the shortening interaction range, proportional to \( L_D \), as \( \gamma \) increases. As \( \gamma \to \infty \), the limiting form for \( \lambda \to 0 \) is likely to be two circular patches joined by a bridge at a distance \( r = O(L_D) \) from the origin.

There is a gradual transition from quasi-elliptical equilibria with \( \lambda \) close to 1 to dumbbell-shaped equilibria for small \( \lambda \). This is quantified in figure 4 by comparing \( \lambda \) with the elliptical aspect ratio \( \lambda_c \) obtained from the second-order spatial moments of the vortex patch. For the barotropic Kirchhoff family of equilibria \( (\gamma = 0) \), each member is an ellipse, hence \( \lambda = \lambda_c \). As \( \gamma \) increases, \( \lambda_c \) peels away from this line at progressively larger \( \lambda \) – this indicates that the vortex is becoming dumbbell-shaped. The family for

![Figure 3](image-url)

**Figure 3.** Selected equilibrium contour shapes for \( \gamma = 0.5 \) (left), 3.0 (middle) and 8.0 (right). In each frame, we show the equilibrium contours for \( \lambda = 0.5 \), for the aspect ratio \( \lambda_c \) at marginal stability, and for the smallest aspect ratio attainable \( \lambda_f \). The plot window is the rectangle \(|x| \leq 2.2, |y| \leq 1.18 \).
\[ \gamma = 0.02, \] which is close to barotropic, evidently exhibits a bifurcation around \( \lambda = 0.21. \) For \( \lambda > 0.21, \) the equilibria are very close to elliptical in shape, but for \( \lambda < 0.21, \) they become dumbbell-shaped, like all the other non-zero \( \gamma \) families. This bifurcation is associated with a known bifurcation occurring in the barotropic family at \( \lambda = \lambda_4 = 2^{1/2} + 1 - 2(2^{1/2} + 1)^{1/2} = 0.216845 \ldots. \) This point coincides with the margin of stability for a wave 4 disturbance (Love 1893, Dritschel 1986). Moreover, Kamm (1987), Cerretelli and Williamson (2003a) and Luzzatto-Fegiz and Williamson (2010) have shown that there are two new branches of equilibria splitting off from the elliptical branch at \( \lambda = \lambda_4. \) One branch is dumbbell-shaped, while the other is eye-shaped (more pointed at the extremities). For non-zero \( \gamma, \) only the dumbbell-shaped equilibria are connected by continuous deformations to circular equilibria at \( \lambda = 1. \) The eye-shaped equilibria presumably still exist, at least for small \( \gamma, \) but they lie on an isolated branch in parameter space. The apparent bifurcation occurring at small \( \gamma \) near \( \lambda = \lambda_4 \) is clearly visible in the particle rotation rate \( \Omega_p \) (the average rate at which fluid particles circulate around the boundary), shown in figure 5 for the four smallest values of \( \gamma \) examined in this study. For \( \gamma = 0.02, \) we see that \( \Omega_p \) strongly dips towards \( \lambda = \lambda_4, \) while as \( \gamma \) increases, \( \Omega_p \) flattens and there is less sign of the bifurcation. Similar sensitivity is seen in the linear stability of the equilibria, discussed further in section 3.2.

Further properties of the equilibria are shown in figure 6 as a function of \( \gamma \) and \( \lambda. \) In figure 6(a) we have the angular impulse \( J = q_0 \iint_D (x^2 + y^2) \, dx \, dy, \) where \( D \) is the region inside the vortex patch, in (b) the total energy \( E = -(q_0/2) \iint_D \psi \, dx \, dy \) (appendix B), in (c) the rotation rate \( \Omega \) and in (d) the particle rotation rate \( \Omega_p. \) Note that the barotropic Kirchhoff family is not represented at \( \gamma = 0; \) rather we use the numerically generated family for \( \gamma = 0.02, \) which is dumbbell-shaped like all other families at small \( \lambda. \) Turning first to the angular impulse \( J \) and energy \( E, \) for every \( \gamma \) considered we find that \( J \) exhibits a maximum at the same point \( \lambda \) where \( E \) exhibits a minimum (this is marked by
By contrast, $J$ and $E$ are monotonic in $\lambda$ for the barotropic Kirchhoff family. If we plot $E$ as a function of $J$, we generally find two branches of solutions joined at a cusp, as illustrated in figure 7 for the $\gamma = 4$ family (all other families are qualitatively similar). Saffman and Szeto (1980) and Saffman (1992) argued that, under these circumstances, the upper branch is linearly stable while the lower branch is unstable (see Dritschel (1985), for why this argument is incomplete). The joint extremum of $J$ and $E$ at $\lambda = \lambda_c$ thus coincides with the margin of stability. Dritschel (1995) has shown that for a pair of liked-signed, unequal-sized barotropic vortices ($\gamma = 0$), the joint extremum of $J$ and $E$ does indeed coincide with the margin of stability. On the other hand, it does not when the vortices are opposite-signed. Notably, at marginal stability $\lambda = \lambda_c$, the linear eigenmode has exactly zero frequency (and growth rate), indicating the existence of multiple branches of equilibria. Here, consistent with this picture, there are two, a lower and an upper branch stemming from a cusp in the $E(J)$ diagram. The same situation occurs for like-signed barotropic vortices, discussed in Dritschel (1995).

The linear stability analysis for QG vortex patches below confirms this behaviour, albeit for only one of the two modes of instability found. This mode has the same symmetry as the equilibria found here, but it is not the first to become unstable for $\gamma \lesssim 3$. Moreover, this mode does not exist for $\gamma = 0$; then the angular impulse varies monotonically with $\lambda$ (for further discussion, see Dritschel 1995, Luzzatto-Fegiz and Williamson 2010).

The rotation rate $\Omega$ of the vortex and the particle rotation rate $\Omega_p$ around its boundary in figures 6(c) and (d) both strongly decrease with increasing $\gamma$ and, to a lesser extent, decrease with decreasing $\lambda$. Note that $\Omega_p \gg \Omega$ for large $\gamma$. In this part of parameter space, the fluid velocity induced by the vortex is confined to a narrow belt of $O(L_D)$ width, and to leading order the velocity on the contour is $\sim q_0 L_D/2$ (Nycander et al. 1993). This implies $\Omega_p \sim q_0 L_D/2P$, where $P$ is the arc length (a circular patch has $P = 2\pi$), a relationship which holds within 3% for $\gamma = 10$. On the other hand, the equilibrium rotation rate $\Omega$ depends on exponentially-small long-range interactions, and is therefore much smaller in magnitude. If we regard the equilibrium as a solitary

![Figure 5](image-url)
wave solution with maximum curvature $K$ then, following Nycander et al. (1993), the predicted value of $\Omega$ is $q_0(KL_D)^3/32$. Figure 8 shows $\Omega\gamma^5$ versus $\gamma = L/L_D$ for $\lambda = 0.5$ (note $q_0 = L = 1$). For this aspect ratio, the equilibria are nearly elliptical, so we can estimate $K \approx \lambda^{-3/2} = 2\sqrt{2}$. This implies $\Omega\gamma^5 \to 1/\sqrt{2}$ as $\gamma \to \infty$, which appears to be a plausible estimate.

3. Linear stability

We next examine the linear stability, to normal-mode disturbances, of the vortex patch equilibria presented in section 2. We briefly review the method used in section 3.1, and then discuss our findings in section 3.2.
Normal-mode linear stability is assessed using the method developed in Dritschel (1995). In that method, disturbances normal to the equilibrium vortex boundary are represented in terms of an angle $\phi$ proportional to the travel time of a particle around the boundary. This simplifies some aspects of the analysis, and improves numerical accuracy. Disturbances are expanded in a truncated Fourier series in $\cos m\phi$ and $\sin m\phi$, retaining wavenumbers $m \leq M = 50$. This results in a $2M \times 2M$ matrix eigenvalue problem for each $\gamma$ and $\lambda$ value considered. We have verified that our results change

![Figure 7](image1)

Figure 7. The total energy $E$ plotted as a function of angular impulse $J$ for the $\gamma = 4$ family of solutions. The smallest $\lambda$ values occur on the lower branch at the bottom right of the figure.

![Figure 8](image2)

Figure 8. The scaled equilibrium rotation rate $\Omega\gamma^3$ versus $\gamma$ for $\gamma = 3–10$, at $\lambda = 0.5$.

### 3.1. Method

Normal-mode linear stability is assessed using the method developed in Dritschel (1995). In that method, disturbances normal to the equilibrium vortex boundary are represented in terms of an angle $\phi$ proportional to the travel time of a particle around the boundary. This simplifies some aspects of the analysis, and improves numerical accuracy. Disturbances are expanded in a truncated Fourier series in $\cos m\phi$ and $\sin m\phi$, retaining wavenumbers $m \leq M = 50$. This results in a $2M \times 2M$ matrix eigenvalue problem for each $\gamma$ and $\lambda$ value considered. We have verified that our results change
insignificantly when doubling $M$, and the results presented below are accurate to within the plotted line width.

### 3.2. Results

The results of the linear stability analysis are presented in figure 9, which shows the growth rates $\sigma_r$ of the two most unstable modes found for the full range of $\gamma$ considered and for $\lambda \leq 0.35$ (stability is found for all $\lambda > 1/3$, the boundary of stability of barotropic Kirchhoff vortices (Love 1893)). The two modes can be broadly identified as a large-$L_D$ mode ($\gamma \lesssim 3$) found on barotropic Kirchhoff vortices, and a distinct small-$L_D$ mode (dominant for $\gamma \gtrsim 3$). The margin of stability $\lambda_c$ falls at the $\sigma_r = 0$ contour in the figure. We see that the margin of stability of the small-$L_D$ mode coincides with the maximum of the angular impulse $J$ and the minimum of the energy $E$ (cf. figures 6(a) and (b)). Here, both the growth rate $\sigma_r$ and the frequency $\sigma_i$ are zero, a so-called “exchange-type” instability (see section 2 for discussion). On the other hand, the margin of stability of the large-$L_D$ mode does not coincide with either the maximum of the angular impulse $J$ or the minimum of the energy $E$. This is despite the fact that both $\sigma_r$ and $\sigma_i$ are zero at $\lambda = \lambda_c$. Here, it is likely that an additional branch of equilibria without two-fold symmetry splits off from the main solution branch, as found in the analogous barotropic Kirchhoff case (Luzzatto-Fegiz and Williamson 2010).

Near the barotropic limit $\gamma \ll 1$, we again see evidence of the bifurcation occurring for $\gamma = 0$ at $\lambda = \lambda_4$ in the growth rates of the unstable modes (figure 10). Until roughly $\lambda = 0.22$, the smallest $\gamma = 0.02$ curve hugs the barotropic one, after which it breaks away (the vortex patch rapidly changes shape through $\lambda = \lambda_4$, cf. figure 4). As $\gamma$ increases, there is a more gradual transition around $\lambda = \lambda_4$. 

![Figure 9. Growth rates $\sigma_r$ of the two most unstable modes in the $\gamma-\lambda$ parameter plane. The contour interval is 0.01.](image)
Figure 11 shows the growth rates as a function of $\gamma$ for three distinct $\gamma$ values corresponding to those illustrated in figure 3. These $\gamma$ values are characteristic representatives of large-$L_D$ behaviour ($\gamma = 0.5$ in the figure), small-$L_D$ behaviour ($\gamma = 8$) and the boundary between the two ($\gamma = 3$). For large $L_D$ or small $\gamma$ (on the left), instability emerges at moderate $\lambda$, then plateaus and finally slightly decreases as $\lambda \to 0$. At small $\lambda$, a second stronger mode emerges and dominates at very small $\lambda$. For intermediate $L_D$ (in the middle), two modes erupt at nearly the same (small) aspect ratio. The first is the large-$L_D$ mode, but this is quickly overwhelmed by the small-$L_D$ mode at smaller $\lambda$. For small $L_D$ (on the right), only the small-$L_D$ mode is seen. However, the instability is substantially weaker than in the other two cases. Again, this is due to the weakening of long-range interactions as $L_D$ decreases.

Finally, it is noteworthy that only two modes of instability occur throughout the entire parameter space (with an exceptional third for very small $\lambda$ and for
some $\gamma$ values).\footnote{These modes, which are much weaker than the primary mode of instability, may be associated with additional turning points in the $E(J)$ diagram, as found by Luzzatto-Fegiz and Williamson (2011) in the barotropic context.} This stands in sharp contrast to the barotropic Kirchhoff case, where there is an infinite sequence of instability modes (at $\lambda = 1/3, 0.2168 \ldots$, etc.) occurring for decreasing $\lambda$ (Love 1893, Dritschel 1986). These modes correspond to wave 3, 4, etc. disturbances, but are evidently not found for $\gamma > 0$, at least for the families of equilibria deformable from circular shapes. The deformation into dumbbell-shaped vortices appears to limit the number of unstable modes (in effect offering greater stability). This appears to explain why we see only the wave 3 instability for small $\gamma$ near $\lambda = 1/3$, something akin to the wave 4 instability for smaller $\lambda$ when vortices are dumbbell-shaped, and no other instabilities.

In summary, smaller $\gamma$ equilibria lose stability at higher $\lambda$ and have higher growth rates than larger $\gamma$ equilibria. This means that smaller vortices, for fixed deformation length $L_D$, destabilise more readily than larger ones.

4. Nonlinear evolution

We next explore the nonlinear evolution of unstable vortex patches near the marginal stability boundary found in section 3. The numerical method is first discussed in section 4.1, and then various forms of nonlinear evolution are illustrated and mapped in the $\gamma-\lambda$ parameter plane in section 4.2.

4.1. Method

We use the contour surgery algorithm (Dritschel 1988, 1989) to study the evolution of the boundary of the vortex patch, including complex processes such as splitting and filamentation. The algorithm solves equation (3) numerically by discretising contours into a finite, variable number of nodes, connected together by local cubic splines. The contour integrations are performed semi-analytically for the singular logarithmic part of the $K_0$ Green function, and by two-point Gaussian quadrature for the nonsingular remainder (Dritschel 1989). Nodes are added, removed and shifted in response to changes in contour curvature. A fourth-order Runge–Kutta time integration method is used with a fixed, standard time step of $\Delta t = 0.025$.

Initially, we start with an equilibrium vortex patch which is slightly disturbed by randomly displacing the $x$ and $y$ coordinates of each node by 1% of $\Delta \phi = 2\pi/n_0$, where $n_0$ is the initial number of nodes (400 for $\gamma \leq 1$ and 800 for $\gamma > 1$). This is done so as not to bias the evolution towards either the symmetric or the asymmetric modes. Thereafter, every 8 times step, the nodes are redistributed using a dimensionless node separation parameter $\mu = 0.2$ and a large-scale length $L_c = P/(\mu n_0)$, where $P$ is the equilibrium contour perimeter. During the evolution (also every 8 time steps), if the distance between two parts of the contour (or between two distinct contours) becomes less than the “cut-off scale” $\delta = \mu^2 L_c/4$, contour surgery is performed (Dritschel 1988). This either splits a contour into two parts or joins two contours together. The settings
for $\mu$, $\delta$ and the frequency of surgery and node redistribution are now standard, and a comprehensive discussion may be found in Fontane and Dritschel (2009).

We evolve the equilibria typically for 100 particle rotation periods $T_p = \frac{2\pi}{\Omega_p}$ (based on the equilibrium value of $\Omega_p$, see figure 6(d)). A few cases were evolved for longer times (for as long as $500T_p$), and no qualitative differences were found. No vortex splitting occurred later than $91T_p$, with splitting times being highest for $\gamma = 2$ and 3, which were both evolved for at least $200T_p$. Due to computational costs, we examine a subset of equilibria having $\gamma = 0.25, 0.5, 1, 1.5$ and 2 to 10 in increments of unity (13 cases in total). We examine the small $\gamma$ cases more closely, as there is a steep decrease in $\lambda_c$ for these values (cf. figure 9).

4.2. Results

Dritschel (1986) and Tang (1987) found that, for the barotropic case ($\gamma = 0$), linear stability of an equilibrium implies nonlinear stability (which, unlike the converse of the statement, is not generally true). For $\gamma > 0$, we find that this statement remains true, within $\Delta\lambda = 0.001$ in almost all cases. The only exception is the $\gamma = 4$ case, where the stability boundary occurs at an aspect ratio smaller by 0.004 than predicted. This may be due to weak vacillations occurring near the margin of stability in this case, as such vacillations are noticeable for $\gamma = 5$ and 6 (see below).

In almost all cases, these weak linear instabilities near $\lambda_c$ nevertheless lead to major structural changes in the vortex shape, from filamentation to splitting. Different types of evolution accompany the two main linear modes discussed above, and there are significant variations in these types. In addition to a visual examination, we quantify area changes at late times to better distinguish these types. The first diagnostic is the fractional change in the total area occupied by the largest one or two vortex patches at the end of the simulation,

$$
\delta A = 1 - \frac{\sum_{i=1}^{2} A_i}{A_0},
$$

where $A_1$ is the area of the largest vortex patch and $A_2$ is the area of the second largest one (if present; otherwise $A_2 = 0$). Here, $A_0 = \pi$ is the area of the original vortex patch. We thus do not include any filamentary debris or smaller vortex patches left over at late times. The second diagnostic is the ratio of vortex areas, $A_2/A_1 \leq 1$, at the end of the simulation.

Three types of instability can be identified by looking at $\delta A$ and $A_2/A_1$ in figure 12. The first, which we call “type 2” (see below for the “type 1” instability), occurs for states with $\gamma < 1$. Here, we see a peak in $\delta A$, and a gradual increase in $A_2/A_1$ from 0. Visual examination indicates that the vortex destabilises asymmetrically, shedding a large filament which may subsequently roll up into a series of smaller vortices. This behaviour was found previously in the barotropic case (Dritschel 1986), apart from the vortex roll up, and is directly associated with the instability of a wave-3 disturbance in this case. An example of this evolution, now for $\gamma = 0.5$ and $\lambda = \lambda_c = 0.296$, is illustrated in figure 13. At late times, a single quasi-elliptical vortex patch remains, whose final aspect ratio is approximately $\lambda = 0.435$ (based on the second-order spatial moments of the patch). This is well within the stable part of the parameter space.
A second type of instability, which we label “type 3i”, is apparent for $1 \leq \gamma < 4$: here the vortex splits asymmetrically with little filamentary debris. An example is illustrated in figure 14 for $\gamma = 2$ and $\lambda = \lambda_c = 0.091$. In fact, there is a smooth transition from type 2 to type 3i, accompanied by a significant growth in the area ratio $A_2/A_1$ and a decay in the area loss $\delta A$, as shown in figure 12. As $\gamma$ increases, the initial filament shed from the vortex increases in size, and has a greater tendency to roll up into a single vortex with little debris. This type of instability reflects a competition between the asymmetric small-$L_D$ linear instability mode, and the symmetric large-$L_D$ mode (cf. figure 9).

Finally, as $\gamma$ increases further, the vortex splits almost perfectly into two identical halves, with negligible filaments. This instability, referred to as “type 3ii”, is illustrated in figure 15 for $\gamma = 10$ and $\lambda = \lambda_c = 0.024$. Such symmetric evolution occurs for $\gamma \geq 4$, and appears directly associated with the symmetric large-$L_D$ linear instability mode. Notice the very slow evolution of the flow, in particular the slow propagation of waves around each vortex boundary (the waves in fact obey a modified KdV equation to leading order in $L_D$, as explained in Nycander et al. 1993).
Figure 14. Type 3i instability, asymmetric split. Here we illustrate the case for \( \gamma = 2 \) and \( \lambda_c = 0.091 \).

Figure 15. Type 3ii instability, symmetric split. Here we illustrate the case for \( \gamma = 10 \) and \( \lambda_c = 0.024 \).
After careful scrutiny, we uncovered an additional type of linear instability which is not apparent from figure 12. In a small range of aspect ratios near marginal stability for $\gamma = 5$ and 6 only, we observed a vacillating state, an example of weakly nonlinear instability. This instability, here called “type 1” as it is the weakest of all, is illustrated in figure 16 for $\gamma = 5$ and $\lambda = 0.024$. We note that $\lambda_c$ attains a maximum near $\gamma = 5$ for the small-$L_D$ linear instability mode (figure 9), but we are not sure of the significance of this for nonlinear vacillation. Vortices exhibiting type 1 instability begin by tilting, similar to that seen at $t = 45.98T_p$ in figure 15, but recover stability by increasing their aspect ratio or widening the bridge between the two halves of the vortex ($t = 23.76T_p$). By angular momentum conservation, this requires the vortex to become more extended. Thereafter, the evolution reverses and the initial vortex shape is recovered by $t = 31.15T_p$. The whole process then repeats itself.

A summary of the types of instability occurring near marginal stability is presented in figure 17. Note the different scales for $\lambda$ in the three panels shown. The barotropic case with $\gamma = 0$ is also shown, following Dritschel (1986) who found the type 2 instability for $\lambda \leq 1/3$. The large-$L_D$ mode found in figure 9 is characterised by the type 2 and 3i instabilities, in which filaments are shed by the equilibrium as it loses stability. The small-$L_D$ mode, on the other hand, has little or no filamentation and results in a split of the equilibrium into two symmetric vortex patches. The transition from one mode to the other is smooth; each consecutive case with larger $\gamma$ having the type 3i instability produces a decreasing amount of filaments and increasingly symmetric vortex patches. There is a division between the types of instabilities which produce filaments and those which do not. Dritschel (1986) found filamentation for the barotropic case, which agrees with our findings for the large-$L_D$ cases. Polvani et al. (1989) observed a suppression of filamentation for large $\gamma$ in the case of the merger of two nearby patches of equal PV. For $\gamma = 1$ they observed a “roll-up” of the filaments, similar to the formation of a small vortex patch observed here at the same value of $\gamma$ (cf. figure 12). For $\gamma = 3$ they noted a complete suppression of filamentation – this is close to the transition from type 3i to type 3ii instabilities observed here (at $\gamma = 3$, we have $\delta A = 0.007$ and $A_2/A_1 \approx 0.93$, together with some weak filamentation, but this is much weaker than at $\gamma = 2$, for which we have $\delta A = 0.017$ and $A_2/A_1 \approx 0.4$). For yet larger $\gamma$, Polvani et al. (1989) observed large-amplitude non-breaking nonlinear waves propagating on the boundary of the vortex, which also closely parallels our observations following a vortex split.
5. Conclusions

This article has examined the form, stability and long-time nonlinear evolution of two-fold symmetric vortex patch equilibria in a single-layer quasi-geostrophic model. The equilibria depend on two parameters: the ratio $\gamma$ of the mean vortex radius $L$ to the intrinsic Rossby deformation length $L_D$, and the ratio $\lambda$ of the minimum to the maximum width of the vortex. The uniform potential vorticity within the vortex may be taken to be unity, as the induced flow is linearly related to potential vorticity.

We have covered a wide range of the $\gamma$–$\lambda$ parameter space in detail, extending a previous study by Polvani et al. (1989). We have furthermore carried out a linear stability analysis to locate the margin of stability to within $\Delta \lambda = 10^{-3}$ over a wide range of $\gamma$ values. Finally, we have examined the nonlinear evolution of marginally unstable vortex equilibria, and associated the types of evolution with the principal modes of linear stability.

There are two principal modes, one occurring for small $\gamma$ which has its origin in the barotropic problem ($\gamma = 0$), and another occurring mainly at intermediate to large $\gamma$, which appears to be unrelated to the barotropic problem. The first “large-$L_D$” mode is asymmetric (at least near the margin of stability), and results in the ejection of a large filament in the nonlinear evolution. The second “small-$L_D$” mode is symmetric, and results in a symmetric split of the vortex patch into two, with negligible
filamentary debris. The nonlinear problem proves even richer, as there is mode competition, leading to asymmetric vortex splits for moderate $\gamma$, specifically $1 \leq \gamma < 4$. Furthermore, we have found a weak nonlinear instability, vacillation, for some aspect ratio values near the margin of stability when $5 \leq \gamma \leq 6$ (these uncertainties arise because we have performed simulations only for integer values of $\gamma$ for $\gamma \geq 1$). This weak instability results in variations of the vortex shape only, with the initial shape recurring from time to time.

Both modes of linear stability exhibit an "exchange-type" instability where both the real and imaginary parts of the eigenfrequency are simultaneously zero at marginal stability. Saffman (1992) argued that such an instability occurs at an extremum of total energy, and Dritschel (1985, 1995) showed that this occurs at joint extrema of both angular impulse and energy (here for fixed $\gamma$) as a function of $\lambda$. This we have verified for the second small-$L_D$ mode, but not for the first large-$L_D$ mode. Notably, the small-$L_D$ mode has the same symmetry as the equilibrium, while the large-$L_D$ mode does not. (There are symmetric modes of instability in the barotropic problem at smaller $\lambda$, the first occurring at $\lambda = 0.2168\ldots$, that exhibit an "exchange-type" instability yet the angular momentum and energy are monotonic functions of $\lambda$ in this case. For further elaboration, see Dritschel (1995) and Luzzatto-Fegiz and Williamson (2010).)

By way of summary, we have found that for a fixed vortex aspect ratio $\lambda$ and Rossby deformation length $L_D$, small vortices are likely to be more unstable than large vortices. Put another way, large vortices can sustain much greater deformations before destabilising than small vortices.

In future work, we plan to generalise this problem in several ways. First, we would like to revisit the two-layer problem studied by Polvani et al. (1989), to provide a deeper and more comprehensive understanding of the properties and stability of vortices, including their long-time nonlinear evolution and transitions into other near-equilibrium forms. Likewise, the two-patch problem, especially for unequal-sized vortices, is highly relevant as indicated by the asymmetric vortex splits observed herein. Asymmetry fundamentally alters the nature of vortex interactions, and notably in the barotropic case, vortex merger is relatively uncommon in the parameter space (Dritschel and Waugh 1992, Dritschel 1995). Similarly, unequal PV in the two patches has received little attention, except in the barotropic case (see Yasuda and Flierl 1995, and references therein). In short, there remain a number of open, fundamental problems in quasi-geostrophic vortex dynamics whose study would help us better comprehend the complex, multi-faceted behaviour of vortices in the atmosphere and oceans.

An important further extension is to study vortex equilibria and their stability in more realistic models of geophysical fluid dynamics. Currently, we are extending the present work to the full gravity-wave permitting shallow-water equations. The results of this will be presented in a forthcoming paper.

References


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Appendix A: Derivation of the linear dispersion relation for small amplitude waves

The following result may be deduced from Waugh and Dritschel (1991).

Consider a circular vortex of radius $r = 1$. The streamfunction is determined from

$$
\frac{1}{r} \frac{d}{dr} \left( r \frac{d\tilde{\psi}}{dr} \right) - \gamma^2 \tilde{\psi} = \begin{cases} 1, & r < 1 \\ 0, & r > 1, \end{cases}
$$

(A.1)

and matching $\tilde{\psi}$ and $\tilde{u}_0 = d\tilde{\psi}/dr$ at $r = 1$, we find

$$
\tilde{\psi} = \begin{cases} -\gamma^{-2} + K_1(\gamma)I_0(\gamma r)/\gamma, & r < 1 \\ -I_1(\gamma)K_0(\gamma r)/\gamma, & r > 1 \end{cases}
$$

(A.2)

and

$$
\tilde{u}_0 = \frac{d\tilde{\psi}}{dr} = \begin{cases} K_1(\gamma)I_1(\gamma r), & r < 1 \\ I_1(\gamma)K_1(\gamma r), & r > 1. \end{cases}
$$

(A.3)

Next, consider an $m$-fold symmetric perturbation to this basic state, having the form

$$
\psi' = \hat{\psi}(r)e^{i(\sigma t - \sigma \theta)}, \quad r' = \hat{r} e^{i(\sigma t - \sigma \theta)},
$$

(A.4)

with $\psi = \tilde{\psi} + \psi'$ and $r = \hat{r} + r'$, where $\hat{r} = 1$ and both $\psi'$ and $r'$ are suitably small. Note that the perturbation satisfies the Helmholtz equation on both sides of the jump since $q' = 0$, there,

$$
(\nabla^2 - \gamma^2)\psi' = 0,
$$

(A.5)

whose solutions are the modified Bessel functions of order $m$:

$$
\psi = \begin{cases} aK_m(\gamma)I_m(\gamma r), & r < 1 \\ aI_m(\gamma)K_m(\gamma r), & r > 1. \end{cases}
$$

(A.6)

Since the vortex boundary moves as a material curve, its radial displacement satisfies

$$
\frac{D\eta}{Dt} = u_r(1 + r', \theta, t) = -\frac{1}{r} \frac{\partial \psi'}{\partial \theta},
$$

(A.7)

which, when linearised, gives simply

$$
\hat{r}(m\hat{\Omega} - \sigma) = -m\hat{\psi}(1),
$$

(A.8)

where $\hat{\Omega} = \tilde{u}_0(1) = K_1(\gamma)I_1(\gamma)$.

Using the continuity of radial and tangential velocities $u_r = u'_r$ and $u_\theta = \tilde{u}_0 + u'_\theta$, respectively, gives $\hat{r} = -a$, and using $\hat{\psi}(1) = aI_m(\gamma)K_m(\gamma)$ gives, after some
Appendix B: Contour-integral form of the energy for quasi-geostrophic vortex–patches

The total energy, kinetic plus potential, of a spatially compact distribution of PV in a single-layer QG flow is given by

\[
E = \frac{1}{2} \int \int (u^2 + v^2 + \psi^2/L_D^2) dx \, dy = -\frac{1}{2} \int \int q \psi \, dx \, dy
\]  

(B.1)

after integrating by parts and using the fact that all fields decay exponentially fast as \( |x| \) and \( |y| \to \infty \). For a single vortex patch, of uniform PV \( q_0 \) in a region \( D \) bounded by a contour \( C \) outside of which \( q = 0 \), we have

\[
E = -\frac{q_0}{2} \int \int_D \psi \, dx \, dy,
\]

(B.2)

where the streamfunction is itself obtained by an integration over the QG Green function:

\[
\psi = -\frac{q_0}{2\pi} \int \int_D K_0(\gamma r) dx' \, dy',
\]

(B.3)

where \( r = |x' - x| \) and \( \gamma = 1/L_D \) henceforth. The purpose of this appendix is to show that the calculation of \( E \) can be reduced to a pair of contour integrals, which proves convenient for its numerical evaluation in the paper.

The starting point is Stokes’ theorem written in polar coordinates:

\[
\int \int_D \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial P}{\partial r} \right) - \frac{1}{r^2} \frac{\partial Q}{\partial \theta} \right] r \, dr \, d\theta = \oint_C P \, dr + Q \, d\theta.
\]

(B.4)

To use this, in \( \psi \) we place \( x \) at the origin of our (polar) coordinate system, so that \( x' - x = r \cos \theta \) and \( y' - y = r \sin \theta \). Then \( \psi \) is given by the integral

\[
\psi = -\frac{q_0}{2\pi} \int \int_D K_0(\gamma r) r \, dr \, d\theta.
\]

(B.5)

This can be reduced to a contour integral by choosing \( Q = 0 \) and \( P = (K_1(\gamma r) + B/r)/\gamma \), for an arbitrary constant \( B \), since then \( r^{-1}d(rP)/dr = -K_0(\gamma r) \). However, to avoid a singularity in the contour integral (and in \( \psi \)), we must take \( B = -1/\gamma \) since \( K_1(z) \sim 1/z \) as \( z \to 0 \). Then, noting that \( r^2 d\theta = (x' - x) dy' - (y' - y) dx' \), and defining the function

\[
H(z) = (zK_1(z) - 1)/z^2,
\]

(B.6)

we obtain

\[
\psi = \frac{q_0}{2\pi} \oint_C H(\gamma r) [(x' - x) dy' - (y' - y) dx'].
\]

(B.7)
Next, we tackle the double integral over $\psi$ needed to calculate the energy $E$:

$$E = -\frac{q_0^2}{4\pi} \oint_C \oint_D H(\gamma r)[(x' - x)dy' - (y' - y)dx'] \, dx \, dy. \tag{B.8}$$

This time, we use polar coordinates relative to a fixed point $x'$ in the outer contour integral, that is $x - x' = r \cos \theta$ and $y - y' = r \sin \theta$ (re-defining the symbol $\theta$). Then, we may write

$$E = +\frac{q_0^2}{4\pi} \oint_C \oint_D H(\gamma r)[r \cos \theta \, dy' - r \sin \theta \, dx'] \, r \, dr \, d\theta. \tag{B.9}$$

This time in Stokes’ theorem, we take $P = 0$ and choose $Q$ either to be $H(\gamma r)r^2 \sin \theta$ or $H(\gamma r)r^2 \cos \theta$, as appropriate. This leads to

$$E = -\frac{q_0^2}{4\pi} \oint_C \oint_D H(\gamma r)r^2[\sin \theta \, dy' + \cos \theta \, dx'] \, dr. \tag{B.10}$$

Then, using $r \cos \theta = x - x'$, $r \sin \theta = y - y'$ and $r \, dr = (x - x') \, dx + (y - y') \, dy$, we arrive at the final form for $E$:

$$E = -\frac{q_0^2}{4\pi} \oint_C \oint_D H(\gamma r)[(x' - x) \cdot dx'][(x' - x) \cdot dx]. \tag{B.11}$$

This can be easily extended to multiple patches by summing over all pairs of associated contour integrals and PV jumps (cf. Dritschel 1985).

It has been verified that this expression gives the correct energy for a circular vortex patch of unit radius, $E = \pi q_0^2[1/2 - I_1(\gamma)K_1(\gamma)]/\gamma^2$, which can be evaluated directly from the form of $\psi$ given in appendix A. Note that $E > 0$ for all $\gamma$ and monotonically decreases to 0 as $\gamma \to \infty$. As $\gamma \to 0$, however, $E \to \infty$. This is perhaps not the result one would expect in this limit, in which the flow is governed by the 2D Euler equations. But energy cannot be defined in this limit, only “excess energy”, by removing a divergent part (Dritschel 1985). Here, using the asymptotic properties of modified Bessel functions (cf. Watson 1966), one can show that $H(\gamma r) \to (1/4)(\ln r^2 - 1) + C + O(\gamma)$ as $\gamma \to 0$, where $C = [\ln(\gamma/2) + \gamma_c]/2$ and $\gamma_c = 0.57721566\ldots$ is Euler’s constant. The leading function of $r$ is exactly that used to compute the excess energy for the 2D Euler equations (Dritschel 1985). The constant $C$ contributes $-\Gamma^2C/2\pi$ to $E$, where $\Gamma = q_0A$ and $A$ is the area of the vortex patch. Hence, $E + \Gamma^2C/2\pi$ — the excess energy — is expected to be finite as $\gamma \to 0$. For a circular patch of unit radius, $\Gamma = q_0\pi$, and $E + \Gamma^2C/2\pi$ reduces to $\pi q_0^2/16$, which is the correct value of the excess energy.

Finally, (B.11) can be generalised to any Green function of the form $G(r)$. Then, the function $H$ is determined from $r^{-1}d(r^2H)/dr = G(r)$ subject to $\lim_{r \to 0}r^2H = 0$. The final expression is the same as in (B.11), omitting the leading $2\pi$ factor ($4\pi$ is replaced by 2).