Revisiting Vacillations in Shallow-Water Models of the Stratosphere Using Potential-Vorticity-Based Numerical Algorithms

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ABSTRACT
Polar vortex vacillations are investigated using long-term simulations of potential-vorticity (PV)-based shallow-water (SW) models for the stratosphere. In the models examined, mechanical forcing is applied through a time-independent topography mimicking tropospheric excitation of the stratosphere. Thermal forcing is applied through a linear relaxation of the mass field to a time-independent equilibrium state mimicking the radiative relaxation taking place in the stratosphere. The SW equations in the PV, velocity divergence, and acceleration divergence representation are solved for a range of resolutions using the “diabatic contour-advective semi-Lagrangian” (DCASL) algorithm and a standard pure semi-Lagrangian (SL) algorithm. Using very different numerical algorithms enables the determination of the degree of numerical sensitivity and the properties of the vacillations with much greater accuracy than in previous related studies.

The focus here is on the Lagrangian or material evolution of the polar vortex. The authors examine quasi-Lagrangian diagnostics based on equivalent latitude, the mass enclosed by PV contours, and the terms involved in its time evolution. The PV field forms the basis for calculating quasi-Lagrangian diagnostics. Variations in the mass enclosed by a PV contour are associated with nonconservative processes such as diabatic heating, friction, and irreversible small-scale mixing. Generally, the mass of the polar vortex increases under the action of diabatic mass fluxes, whereas it decreases under the action of dissipative mass fluxes.

The results herein differ from previous results reported at T42 resolution by Rong and Waugh in which a spectral transform algorithm is used to solve the SW equations in a vorticity–divergence–mass representation, and in which dissipation is provided by explicitly damping vorticity using hyperdiffusion. Except for the first large-amplitude oscillation, there is little sign of a clear, systematic phase shift between the dissipative and diabatic mass fluxes across the edge of the polar vortex, as proposed by Rong and Waugh as the main mechanism responsible for the vacillations. Concomitant with the absence of a phase shift, the vacillations tend to decay and occur intermittently. Rather than a phase shift, inherent fluctuations in both the diabatic and mass fluxes across the edge of the polar vortex appear to be responsible for the vacillations.

1. Introduction

Various numerical models (see, e.g., Matsuno 1971; Holton and Mass 1976; Scott and Polvani 2006) have been used to simulate the internal variability of the stratosphere, which is thought to have an important impact on surface weather and climate. In these numerical models, a common device to induce variability is the inclusion of either time-independent or time-dependent topographic forcing respectively to exclude or include the variability exerted by the troposphere. A particular feature studied by the numerical models using time-independent topographic forcing is the development of vacillations between westerlies and easterlies at high latitudes. The picture emerging from the observed data analysis and
the three-dimensional numerical simulations is that these polar vortex vacillations take place in the upper levels of the stratosphere with a subsequent downward progression of the anomalies in such a way that the zonal-mean zonal wind anomalies may even reach the lower stratosphere at high latitudes. The vacillations of the vortex are believed to result from the out-of-phase action of the radiative cooling and the Rossby wave forcing as, respectively, the strengthening and weakening factors for the polar vortex. In this picture, the out-of-phase action is provided by the inhibition/facilitation of the vertical propagation of Rossby waves from the lower to the upper stratosphere (Scott and Polvani 2006).

A complementary picture has recently been presented by Rong and Waugh (2004, hereafter RW04) using a pseudospectral shallow-water (SW) numerical model in a vorticity–divergence–mass representation with hyperdiffusion applied to the vorticity field. In the absence of vertical structure, for sufficiently strong topographic forcing and at T42 resolution, regular vacillations between the westerlies and easterlies at high latitudes corresponding to regular cycles of vortex recovery and breakdown are obtained. The horizontal structure of the polar vortex thus seems to play at least some part in generating internal stratospheric variability. In this complementary picture, the competition is between radiative and dissipative effects whose out-of-phase action is involved in the transition between westerlies and easterlies. RW04 argue that the vacillations can be thought of as oscillations between states with strong and weak potential-vorticity (PV) gradients. The transitions required from strong-to-weak and weak-to-strong PV gradients are caused by, respectively, enhanced dissipation due to Rossby wave breaking and radiative effects. Despite the plausibility of this picture, there are indications from the higher-resolution T85 simulations of RW04 for strong topographic forcing that neither the regular vacillation nor the mechanisms outlined above survive when the resolution is increased. Moreover, the sensitivity is not restricted to resolution. There is already an indication of flow sensitivity in the low-resolution T42 simulation of RW04: the regular vacillations cease to exist at sufficiently large times when a very strong vortex forms apparently due to the spurious effects of hyperdiffusion.

Resolution and flow sensitivities, as well as the lack of confidence in hyperdiffusion, have led us to reexamine the nature of vacillations and their underlying mechanisms by making use of two PV-based numerical models: (i) the “diabatic contour-advective semi-Lagrangian” (DCASL) algorithm capable of exceptionally high Lagrangian resolution of PV and (ii) a pure semi-Lagrangian (SL) algorithm for PV at moderate to high Eulerian resolutions (Mohebalhojeh and Dritschel 2009). In addition to the standard experiment of RW04, a new set of experiments is designed to explore flow sensitivity.

The rest of this paper is organized as follows. In the next section, the model and diagnostics are introduced. The SW model employed here, using a PV, velocity divergence, and acceleration divergence formulation, achieves higher-order accuracy than the previous SW models (e.g., RW04). Section 3 is devoted to the analysis of the results. Finally, some concluding remarks are given in section 4.

2. Model and diagnostics

The SW equations can be written in a PV, velocity divergence, and acceleration divergence representation as

\[
\frac{DQ}{Dt} = S_Q = -\frac{Q}{1 + \tilde{h}} S_h, \tag{2.1}
\]

\[
\frac{\partial \tilde{h}}{\partial t} = \gamma - 2 \left[ \frac{\partial u}{\partial \tilde{h}} \frac{\partial \tilde{u}}{\partial \tilde{h}} + \tilde{\zeta} \right] + \frac{\partial v}{\partial \tilde{h}} \left( \frac{\partial \tilde{v}}{\partial \tilde{h}} - \delta \right)
- \nabla \cdot (\tilde{h} \nabla), \tag{2.2}
\]

\[
\frac{\partial \gamma}{\partial t} = c^2 \nabla^2 \left\{ \nabla \cdot [(1 + \tilde{h}) V] - S_h - \frac{\partial \tilde{h}}{\partial t} \right\}
+ \frac{2\Omega E}{a^2} \frac{\partial B}{\partial \lambda} - \nabla \cdot (ZV), \tag{2.3}
\]

where \(S_h\) is the source for the nondimensional perturbation depth \(\tilde{h}\). Following RW04, we use Newtonian cooling so that

\[
S_h = \frac{(H_f/H_0) - 1}{\tau} + \frac{(H_f/H_0)\tilde{h} - \tilde{h}}{\tau}. \tag{2.4}
\]

Above \(Q = (f + \tilde{\zeta})/(1 + \tilde{h})\) is the Rossby–Ertel potential vorticity, \(\delta\) is the velocity divergence, \(\gamma = f\tilde{\zeta} - c^2 \nabla^2 (\tilde{h} + h) - \beta u\) is the acceleration divergence, \(\tilde{\zeta}\) is the relative vorticity, \(f\) is the Coriolis parameter, \(\beta\) is the northward gradient of \(f\), \(v\) is the velocity vector, \(u\) is the velocity component in the longitudinal (\(l\)) direction, \(v\) is the velocity component in the latitudinal (\(\phi\)) direction, \(h = H_0/(1 + \tilde{h})\) is the depth field, \(h_c\) is the equilibrium depth field, \(h_b\) is the bottom topography, \(S_h\) is the source term for \(h\), \(S_Q\) is the source term for \(Q\), \(H_0\) is the global mean depth at initial time \(t = 0\), \(H_c\) is the global mean equilibrium depth, \(\tau\) is the relaxation time, \(S_h\) is the source term for \(h\), \(Z = f(f + \tilde{\zeta})\), \(B = c^2(\tilde{h} + h_b) + (1/2)|v|^2\) is the Bernoulli pressure, \(a\) is the earth’s radius, \(g\) is the acceleration due to gravity, \(c\) is a gravity wave speed, and \(D/Dt = \partial/\partial t + v \cdot \nabla\) the material derivative. The horizontal lengths are nondimensionalized by the earth’s
radius $a = 6.371 \times 10^6$ m and time by one day $T_{\text{day}} = 2\pi/\Omega_E$, with $\Omega_E = 7.292 \times 10^{-5}$ s$^{-1}$. The nondimensional $c$ is equal to $\sqrt{gH_0(T_{\text{day}}/a)} = 4.25$, where $g$ and $H_0$ are set to 9.806 m s$^{-2}$ and 8 km, respectively. By the nondimensionalization used, we can set $a = 1$, $\Omega = 2\pi$, $f = 4\pi \sin \phi$, and $\beta = 4\pi \cos \phi$ in (2.2) and (2.3) as well as in the definition of the prognostic variables $Q$, $\delta$, and $\gamma$.

The topographic forcing takes the same form used by RW04, namely

$$h_b = \frac{H_b}{H_0} (1 - e^{-t/\tau_b}) \cos \phi \cos[(\phi - \phi_b)/\Delta\phi]^2,$$

(2.5)
in which $H_0$ is the dimensional amplitude of topography, $\phi_0 = 45^\circ$ is the central latitude of the topographic forcing, $\Delta\phi = 14.14^\circ$, and $\tau_b = 5$ days.

To gain an understanding of the zonal wind vacillations, a variety of diagnostics are calculated and analyzed. Eulerian diagnostics are provided by the zonal-mean zonal momentum equation

$$\overline{u}_t + \overline{u}^2 + \frac{1}{h \cos \phi} \nabla \cdot \mathbf{F} - \frac{ghh'\overline{h}a}{ah \cos \phi} - \frac{1}{h}(h'\overline{u}'),$$

$$+ \frac{1}{h} \overline{u}' \overline{H}' + X^*,$$

(2.6)

where $\mathbf{F} = [1/(a \cos \phi)](F^{(\phi)} \cos \phi)$ and $F^{(\phi)} = -(hv')/u'$ is the SW version of the Eliassen–Palm (EP) flux and $X$ represents the nonconservative forcing contributed by numerical dissipation. Here an overbar represents the zonal mean, and an asterisk together with an overbar is a mass-weighted zonal mean. The second term on the right-hand side is the topographic forcing that acts to provide the wave-generating mechanism. The third and fourth terms on the right-hand side represent the eddy transport of mass and the mass source, respectively. It is interesting to note that (2.6) is nothing but the SW form of the transformed Eulerian mean zonal momentum equation in isentropic coordinates for the primitive equations (Andrews et al. 1987).

Although useful, the Eulerian zonal-mean diagnostics show only a limited power in capturing the dominant features of the polar vortex when it becomes highly nonzonal, a situation that is not uncommon in the winter stratosphere. In such cases, the so-called quasi-Lagrangian diagnostics based on the distribution of PV and mass following the motion can give us valuable information about the evolution of the vortex.

The area enclosed by PV contours is one of the key quasi-Lagrangian diagnostics. It is expressed as the equivalent latitude $\phi_{\text{equiv}}(Q) = \sin^{-1}[1 - A(Q)/(2\pi a^2)]$, with $A(Q)$ being the area of the PV contour (the area of the domain with PV greater than $Q$). The vortex evolution can be described using the equivalent latitude and the wind speed averaged along the $Q = 2.2 \times 10^{-8}$ m$^{-1}$ s$^{-1}$ contour level as representative of the edge of the polar vortex. Note that the latter quantity, denoted by $|\mathbf{v}_{\text{edge}}|$, is different from the Lagrangian measure of jet strength defined by RW04 as the maximum over all contour levels of the contour mean wind speed.

Using PV contours, we calculate modified Lagrangian mean (MLM) diagnostics of the flow (see also Sobel and Plumb 1999; Thuburn and Lagneau 1999; Neu 2002). The mass within the PV $= Q$ contour is given by $M(Q, t) = \int_{PV > Q} \rho h(Q, t) \, dA$, where $dA$ is the area element in spherical coordinates. The mass tendency is expressed as

$$\frac{\partial M}{\partial t} = \int (1 + \dot{h}) \frac{DQ}{dt}_{\text{ad}} \, dA - Q \int \frac{H_0S_h}{VQ} \, dl + \int \int_{PV > Q} H_0S_h \, dA,$$

(2.7)

where $DQ/dt_{\text{ad}}$ denotes the adiabatic material change of PV caused by numerical dissipation. In the right-hand side of (2.7), the first term corresponds to the mass flux across the contour due to dissipative processes, the second term to the mass flux across the contour due to diabatic processes, and the last term to the mass source/sink within (north of) the contour. For our PV-based algorithms and in particular for the DCASL algorithm, the direct computation of $DQ/dt_{\text{ad}}$ and thus the dissipative flux is almost impractical. Therefore, in the same way as in Thuburn and Lagneau (1999), the dissipative flux is estimated as a residual in (2.7) by a finite-difference evaluation of the mass tendency $\partial M/\partial t$. In the computations examined in this paper the contour integral of the mass sink over the vortex is small in comparison with the others, and the change in mass is due primarily to the combined effects of the diabatic and dissipative mass fluxes.

For a full description of the DCASL and SL algorithms used, one may consult Mohebalhojeh and Dritschel (2009), where two types of DCASL algorithms as well as a fully Lagrangian DCASL algorithm are introduced and compared in detail. In addition to what has already been reported in Mohebalhojeh and Dritschel (2009), the current study rests on our examination of the regularization properties of the recontouring process used in the DCASL algorithms and the performance of the DCASL and SL algorithms in an idealized experiment. The experiment involves advecting a patch of uniform PV in the presence of time-varying forcing where the exact solution is available in both Lagrangian and Eulerian representations. A full examination of the regularization properties of this experiment is beyond the scope of the
present paper. The main finding, however, is that by using multiple PV sets with successively finer PV jumps in the DCASL algorithm, one can make the recontouring a regularization process and reduce the amplitude of the grid-based part of the solution for PV. This proves helpful in dealing with nonsmooth and strong forcing as is the case here. The regularization ensures that the Lagrangian representation is smoothed, and thus not overrepresented, by recontouring.

The DCASL and SL algorithms differ in the representation of PV due to the distinct way they dissipate PV. In the grid-based SL algorithm, PV is dissipated due to the bicubic Lagrange interpolation used, which leads to a biharmonic diffusion with a coefficient proportional to the fourth power of the spatial grid size (Durran 1999). In this way, the SL algorithm damps both finescale filamentary structures and sharp-edged vortices. In the DCASL algorithm, the contour-based part of PV is smoothed by surgery and recontouring at, respectively, 0.1- and 0.8-day time intervals, and the small grid-based part of the PV is smoothed as in the SL algorithm. Because this grid-based part is generally a very small fraction of the total (often less than 1%), dissipation in the DCASL algorithm is primarily due to surgery and recontouring. The use of surgery makes the DCASL algorithm distinct from the SL algorithm in its ability to maintain sharp-edged vortices. There is one more aspect of PV dissipation that is worth emphasizing before going further: the difference between our PV-based algorithms and the spectral-transform algorithm employed by RW04 that solves the SW equations in vorticity, divergence, and mass variables and explicitly damps vorticity using hyperdiffusion. In the latter, PV is dissipated by the \((\nu/h)^2\partial z\) operator, where \(\nu\) is the hyperdiffusion coefficient. To compare with the form of diffusion present in the SL algorithm, we write this operator as \((\nu/h)^2\partial z(hQ) + (\nu/h)f\).

In the limit of vanishing perturbation depth, the first term takes the form of PV hyperdiffusion \(\nu\partial zQ\) while the second term becomes a zonally independent source for PV.

For the long-term simulations needed for this study, computational considerations led us to employ the type-I DCASL algorithm of Mohebalhojeh and Dritschel (2009), which, apart from the introduction of multiple PV sets, is essentially the algorithm of Dritschel and Ambaum (2006) that makes use of both contour and grid representations. In this paper, we simply use the term DCASL to refer to the type-I DCASL algorithm with three PV sets. For the first, second, and third PV sets, the non-dimensional PV jumps are given the values \(\pi/16, \pi/256,\) and \(\pi/1024,\) respectively. The small size of the PV jumps can be appreciated by noting that, for the experiments here, the upper bound for the absolute value of the non-dimensional PV is about 16\(\pi\).

3. Numerical results

a. The standard experiment

We take the large-amplitude topographic forcing case of RW04 with \(H_B = 3000\) m and relaxation time of \(\tau_e = 10\) days as our standard test case. Using PV-based algorithms at higher Eulerian–Lagrangian resolution allows a reassessment of the nature and robustness of zonal vacillations. Further, our standard test case uses exactly the same initial conditions as in Fig. 1 of RW04—that is, a zonal jet characteristic of the winter middle stratosphere taken from Polvani et al. (1995) in balance with the layer thickness \(\bar{h}\). The equilibrium depth field is also a slightly modified version of that used by Polvani et al. (1995). For reference, spatial resolution is denoted by \(N \times N\), with \(N\) being the number of grid points in latitude and longitude. In this way, the latitudinal resolution becomes twice that of the longitudinal resolution to compensate for the lower formal order of accuracy of finite differencing in latitude compared to spectral transforms in longitude (Mohebalhojeh and Dritschel 2007).

The numerical experiments have been carried out with successive doublings of resolution for the SL algorithm with \(N = (64, 128, 256, 512, 1024)\) and the DCASL algorithm with \(N = (128, 256, 512)\) for 1000 days, but the focus here will be on the first 400 days. In addition to providing a point of comparison with the main body of results in RW04, this time span is sufficiently long to enable us to address the questions surrounding vacillations without involving uncertainties in the flow evolution on much longer time scales. Unless stated otherwise, diagnostics are presented for the SL algorithm at 256 \(\times 256\), 512 \(\times 512\), and 1024 \(\times 1024\) resolutions and the DCASL algorithm at 256 \(\times 256\) resolution.

To start off, our Fig. 1 shows the first 400 days of the evolution of the zonal-mean zonal wind \(\bar{u}\) as the simplest Eulerian diagnostic. The presence of the cycles of westerlies and easterlies at high latitudes and the development of an easterly jet in the tropics are evident. However, in clear contrast with the low-resolution T42 results of RW04 reported in their Fig. 4a, there is no sign of regularity or reproducibility of the vacillation cycles.

Neither the number of the cycles nor the period of each cycle is reproducible across the four results shown here and that in Fig. 4a of RW04. A particularly notable feature is the presence of short-period vacillations, examples of which can be seen between days 50 and 100 of the DCASL simulation and around day 200 of the SL simulation at 1024 \(\times 1024\) resolution, alongside longer period vacillations. This single observation points to loss of coherent variability. Disregarding the regularity, the Eulerian diagnostics coming from the zonal-mean zonal momentum equation are in agreement with the results...
of RW04. The results indicate that the divergence of horizontal momentum flux, the topographic forcing, and the Coriolis torque are the dominant factors determining the zonal-mean zonal momentum time evolution. The cycles of the zonal wind are due to subtle changes in the balance among these terms. Therefore, henceforth we focus entirely on Lagrangian diagnostics. It should be noted that the resolutions given here do have only an Eulerian sense. DCASL offers a much higher Lagrangian resolution. To illustrate, Fig. 2 shows the PV field during the early evolution from the strong vortex stage at \( t = 20 \) days to the weak vortex stage at \( t = 60 \) days. For this flow, \( t = 20 \) days is still within the interval of deterministic predictability and convergence in an initial-value sense. This can be seen in the way that the SL results converge when we go to higher resolution and perhaps more remarkably the way the 1024 \( \times \) 1024 SL PV field converges to the 256 \( \times \) 256 DCASL one. The opposite is true for \( t = 60 \) days, when both deterministic predictability and convergence are lost. As we shall see shortly, \( t = 60 \) days corresponds approximately to the time when the vortex is weakest during the first vacillation cycle. Overall, in our higher-resolution results for the PV-based algorithms, the evolution of PV is faster compared with the low-resolution T42 results of RW04. In addition to demonstrating numerical convergence, the figure clearly shows that the vortex breakdown is nowhere near complete. This is in sharp contrast with the nearly complete breakdown seen in Fig. 3c of RW04 at \( t = 80 \) days (see also their Fig. 6). This contrast is crucial in our quest to understand the nature of vacillations.

Figure 3 shows the evolution of the equivalent latitude for PV contours \( \phi_{\text{equiv}}(Q) \), together with \( |v_{\text{edge}}| \). The thick solid curves show the equivalent latitude of the PV \( = 2.2 \times 10^{-8} \) m\(^{-1}\) s\(^{-1}\) contour, which lies within the vortex edge region. At \( t = 0 \) the polar vortex extends from the North Pole to 70°N and \( |v_{\text{edge}}| \) is about 40 m s\(^{-1}\). In the first few days, the dominance of thermal forcing over dissipation leads to an increase in the latitudinal spread of the vortex and an increase in \( |v_{\text{edge}}| \). With the buildup of the topography, however, this early development is reversed to a gradual erosion of the vortex. In all of the results shown, the amplitude of the vacillations decays with time. Further, except for the SL simulation at the lowest resolution of 256 \( \times \) 256, the polar vortex never recovers its full extent at the end of the early development mentioned above. The number and duration of the cycles vary between the DCASL and the SL simulations, as well as across resolution, again pointing to the lack of coherent variability. There is also a clear sign that the equilibrium state to which the flows approach is different in the DCASL and the SL simulations. The mean wind speed around the edge in the SL simulations relaxes to a much lower value than in the DCASL simulation, where there is a gradual increase in
accompanied by irregular oscillations after the first recovery.

The next revealing diagnostic shown in Fig. 4 is the evolution of the mass within PV contours. It should be noted that, for ease of comparison, the mass values and the terms in (2.7) have been scaled by the square of the earth radius. As expected, a close agreement can be seen between the evolution of the mass within the polar vortex in Fig. 4 and the area of the polar vortex and \( |v_{\text{edge}}| \) in Fig. 3. The decay in amplitude of vacillations with time and the irregularity and irreproducibility of the vacillations across the algorithms and resolutions show up as the common feature. Overall, the signature of cycles of vacillations is stronger in this diagnostic. It is the first cycle during the first 100 days that always reaches the largest amplitude.

Figure 5 shows the evolution of the terms involved in (2.7) for the mass tendency. The results shown are for the DCASL algorithm and include the mass tendency \( \partial M/\partial t \), the dissipative flux \( \int (1 + h) DQ/D t |_{v_{\text{ad}}} |VQ| dl \), the diabatic flux \( -Q \int H_{0} S_{x}/|VQ| dl \), and the mass source within PV contours \( \int_{\text{PV} Q} H_{0} S_{x} dA \). For ease of comparison, the time evolution of the mass within PV contours is replotted in Fig. 5a. As in RW04, the mass source contribution to the time tendency is negligible. As the dominant terms, the diabatic and dissipative fluxes result in, respectively, an increase and a decrease in the mass enclosed by the polar vortex. In clear contrast with the T42 results in Fig. 5 of RW04, our results in Fig. 5b show little sign of the out-of-phase variation of the diabatic and dissipative mass fluxes. It seems that the opposite is commonplace. That is, the two fluxes usually increase or decrease simultaneously. Instead of the out-of-phase variations, the amplitude variations of the fluxes seem to generate the cycles. In particular, the diabatic mass flux shows a greater amplitude variation. Acquiring its maximum value around time \( t = 570 \) days, its amplitude undergoes a general decrease overlaying irregular oscillations.

The phase portrait of the mass enclosed by the polar vortex and its time tendency (Fig. 6) gives us a dynamical system view of the evolution of the polar vortex. For clarity, subsequent points have not been connected. The absence of a regular, repeatable cycle is evident for both the DCASL and SL results. Except at the extreme right of the portrait associated with the early evolution of the flow, what is seen is a nearly random scattering of points. It is also worth pointing out that by increasing resolution, the SL algorithm becomes closer to the DCASL algorithm both in terms of the minimum value of the mass and the extrema of the mass tendency.

To better understand the flow variability, we follow Scott and Polvani (2006) by examining frequency spectra,
though here for the Lagrangian time series of the mass within and the diabatic and dissipative fluxes across the edge of the polar vortex (Fig. 7). Regularity of the variation of the Lagrangian diagnostics should be reflected by narrow peaks in the spectra. Figure 7 shows that there are no such narrow peaks in all of the resolutions, and for both numerical methods.

Although not shown for brevity, the results of the SL algorithm at $64 \times 64$ and $128 \times 128$ resolution and the DCASL algorithm at $128 \times 128$ and $512 \times 512$ resolution corroborate the picture provided by the foregoing diagnostics at the resolutions presented. What is worth mentioning is the absence of regular vacillations in any of the resolutions examined, and in particular in the SL algorithm at $64 \times 64$ and $128 \times 128$ resolution, which are comparable to T42 spectral resolution in terms of the number of grid points in, respectively, latitudinal and longitudinal directions [see Table 1 in Mohebalhojeh and Dritschel (2007)]. This provides a further departure from the spectral transform results of RW04 that indicate a transition from regular vacillations at T42 to irregular vacillations at T85.

The mean stratospheric jet may not be the best state to start from as it may precondition the flow for vacillations. In the next two subsections, therefore, we present results for two opposite states in terms of smoothness of flow and complexity of PV distribution.

b. Starting from a state of rest

A state of rest provides the simplest setup to see if regular vacillations, as manifestations of coherent variability, can naturally arise and survive. The quasi-Lagrangian diagnostics associated with the mass within PV contours are presented in Fig. 8. As the evolution of the diabatic mass flux shows, the initial state is largely in radiative disequilibrium, leading to the formation of a number of cycles, though irregularly, with the largest amplitude early in the flow evolution. The variations of the diabatic mass flux, however, are largely canceled out by the in-phase variations in the dissipative mass flux. What we have in Fig. 8a is the clear signature of a damped oscillator and not a regular vacillator. This picture is strongly confirmed by the behavior of the mean absolute value of the PV gradient across the edge of the polar vortex shown in Fig. 9b, where the initial overshoot and undershoot are seen to give way to a state free of any significant variation of note. This is despite the fact that the stretching deformation remains active in changing the contour length and in preparing the grounds for the action of dissipation. This can be seen, for example,
around \( t = 250 \) days where both the contour length and the dissipate flux have large values near their corresponding maxima.

c. Starting from a fully developed state

To gain insight into the role of the initial PV gradient across the edge of the polar vortex, here we start the experiment from a fully developed state. We thus need a diabatic PV inversion to recover the other fields at the initial time and to take into account the effect of thermal relaxation. Although more accurate PV inversion procedures can be used, for our current purpose the Bolin–Charney PV inversion provides sufficient accuracy.

Let us briefly introduce the PV inversion relations involved. The Bolin–Charney balance relations are obtained by setting to zero the variable \( J \) and its first time derivative, where \( J \) is defined by

\[
J = \frac{\mathbf{f} z}{\alpha \phi} + \frac{\mathbf{v} v}{\alpha \phi} \tag{3.1}
\]

In (3.1), \( \mathbf{v} \) is the rotational velocity, \( \psi \) denotes the streamfunction, \( u_\phi \) and \( v_\phi \) are respectively the longitudinal and latitudinal components of \( \mathbf{v}_\phi \), and \( \mathbf{z} \) is the unit vector in the local vertical direction. The balance relation \( J = 0 \) together with the definition of PV and the Poisson equation \( \nabla^2 \phi = \zeta \) constitutes a closed system of equations to solve for \( \mathbf{v}_\phi \) and \( \mathbf{v} \) from the known instantaneous distribution of PV. The balanced divergence \( \delta \) is obtained from \( \frac{\partial \mathbf{v} \cdot \nabla \zeta}{\partial t} = 0 \), which gives

\[
\left( c^2 \nabla^2 - \mathbf{f}^2 \right) \delta = \mathbf{f} \mathbf{v} \cdot \nabla \zeta - c^2 \mathbf{v} \cdot \mathbf{v} \cdot \left( \mathbf{f} \nabla \zeta - S \right) + f \beta \mathbf{v}
\]

\[
+ \beta \frac{\partial u_\phi}{\partial t} + 2 \frac{\partial}{\partial \phi} \left( \frac{\partial u_\phi}{\partial \phi} \left( \frac{\partial u_\phi}{\partial \phi} + \zeta \right) + \left( \frac{\partial v_\phi}{\partial \phi} \right)^2 \right)
\]

\[
+ 2 \left( u_\phi \frac{\partial u_\phi}{\partial t} + v_\phi \frac{\partial v_\phi}{\partial t} \right), \tag{3.2}
\]

where

\[
\frac{\partial \mathbf{v}_\phi}{\partial t} = \mathbf{z} \times \nabla \mathbf{v}^2 \left( - \mathbf{v} \cdot \left( \mathbf{f} + \zeta \right) \mathbf{v} \right). \tag{3.3}
\]

Using (3.3), the Helmholtz decomposition \( \mathbf{v} = \mathbf{z} \times \mathbf{v}_\psi + \mathbf{V}_\psi \) and the Poisson equation \( \nabla^2 \chi = \delta \), (3.2) can be solved to determine \( \delta \) and \( \mathbf{v} \).
We can now take the PV distribution at any given start time \( t_s \) from the standard experiment, invert it by using the diabatic Bolin–Charney balance relations, determine a new initial state, and carry out a new set of integrations. For \( t_s = 100 \) days, the corresponding results for the terms involved in the evolution of the mass enclosed by PV contours are shown in Fig. 10. Note that \( t = t_s \) is the time when the mean PV gradient across the edge of the vortex is close to its time average in the standard experiment. The results are obtained using the DCASL algorithm. Compared with the previous results that start from a smooth flow, the reduction observed in the amplitude of vacillations is striking. This is particularly true for the first 100 days of integration where there is no sign of vacillations. Moreover, the vacillation cycles randomly occurring later during the evolution are never comparable to the first large-amplitude vacillation in the previous experiments. From the start, the diabatic and dissipative fluxes vary in phase and largely cancel each other. To better understand the large differences we observe in behavior between this and the previous sets of experiments, Fig. 11 shows the time evolution of (a) the total absolute value of PV gradient \( \int |VQ| \, dl \), (b) the mean absolute value of PV gradient \( \int |VQ| \, dl/\int dl \), and (c) the length \( \int dl \) of the contour \( Q = 2.2 \times 10^{-8} \text{ m}^{-1} \text{ s}^{-1} \) at the edge of the polar vortex. The common feature of all three curves is the presence of fluctuations on a variety of time scales that seem to stay around a constant mean value for about the first 200 days. For the next 200 days, however, the mean value undergoes a decrease for the contour length and an increase for the mean absolute value of PV gradient. Incidentally, this is the same period during which the amplitude of vacillations shows an increase in Fig. 10a for the mass within the vortex.

We have repeated the procedure above for two other choices of starting time, \( t_s = 0 \) and \( t_s = 50 \) days. The first choice adds the diabatic PV inversion at the initial time to the standard experiment. The second choice is for a time in the standard experiment when the mean gradient is at its peak during the first cycle. For brevity, we mention only the main findings of the latter two experiments, that is, (i) the lack of sensitivity to the exact form
of initialization and (ii) the controlling role of the mean PV gradient across the edge of the vortex. The former is inferred by the statistically similar solutions for the experiment with the starting time of \( t_s = 50 \) and the standard experiment. The latter is inferred by noting that the results for the starting time of \( t_s = 50 \) days exhibit damped oscillations similar to the standard experiment and to that starting from a state of rest. One can conclude that whenever the mean PV gradient across the edge of the vortex deviates substantially from its near-equilibrium value, an oscillation follows.

d. Summary and discussion

A particularly useful way of summarizing the complementary views to vacillations given by the foregoing results for the three sets of experiments is to look at the autocorrelation and cross correlation for the mass within, and the diabatic and dissipative mass fluxes across the edge of the polar vortex (Fig. 12; all results are for the DCASL algorithm). The mass autocorrelation of the standard experiment exhibits the signature of a damped oscillator. The same can be said for the experiment starting from a state of rest where the damping around zero time lag is sharper and there is also a tail of small-amplitude activity. An even sharper and more localized damping is observed when starting from the developed state \( t_s = 100 \) days for which there is a distinctive tail at time lags greater than 200 days. The latter tail can be attributed to the intermittent nature of vacillations. For the diabatic and dissipative mass fluxes, from its peak at zero time lag, the autocorrelation decays rapidly to near-zero values, indicating the noisy nature of both fluxes. This is particularly true for the diabatic flux in the experiments starting from a state of rest and from a fully developed state (\( t_s = 100 \) days). Although less obvious, there is a weak oscillatory behavior whose signature is different between the two fluxes. The most important piece of information comes perhaps from the cross correlation between the dissipative and diabatic fluxes. It peaks very close to zero time lag, indicating the dominantly in-phase variation of the two fluxes. The fact that the signature of oscillatory behavior is different between the dissipative and diabatic fluxes manifests itself in rapid variations of the cross correlation. This is particularly true for the experiment starting from a fully developed state.

For the standard experiment, the same autocorrelation and cross-correlation diagnostics carried out for the SL algorithm at \( 256 \times 256, 512 \times 512, \) and \( 1024 \times 1024 \) resolution give results in overall agreement with those shown above for the DCASL algorithm. This is despite the fact that the DCASL and SL algorithms are radically different in the way they represent PV, particularly in their ability to maintain sharp-edged vortices. Therefore, the damped nature of oscillations is insensitive to the form of PV dissipation.

A reviewer has raised the question of whether a larger topographic Rossby wave forcing can completely disrupt the vortex and bring in regular oscillations by increasing dissipation in the DCASL simulations.
answer this, the standard experiment and all of the diagnostics have been carried out for the DCASL algorithm at 256 × 256 resolution but with $H_b = 3500–6000$ m in 500-m intervals. For brevity, only the evolution of the mass within the edge of the polar vortex is presented here for $H_b = (3000, 4000, 5000, 6000)$ m over a longer span of 1000 days (see Fig. 13). As expected, as $H_b$ increases, the larger stirring of the flow leads to a larger dissipation and disruption of the vortex over the first breakdown. The small-amplitude oscillations that follow the first breakdown are evidently irregular for all cases and either intermittent or damped for each of the experiments with the topographic forcing larger that the main case examined in the paper. The case with $H_b = 4000$ m seems to exhibit the most regular and lasting oscillations over the whole span of simulation. Let us have a closer look at this case in Fig. 14, presenting the evolution of the terms involved in (2.7) for the mass tendency. While the first breakdown at around time $t = 50$ days is consistent with a phase shift between the dissipative and diabatic mass fluxes, the remaining noticeable changes in the mass within the edge of the polar vortex are largely due to fluctuations in the amplitude of either the diabatic forcing or the dissipation. A clear example of this is the mass gain leading to the subsequent partial recovery around time $t = 75$ days after the first breakdown. A fluctuation in the amplitude of the diabatic mass flux is responsible for the mass gain. There is clearly an in-phase fluctuation in the amplitude of the dissipative mass flux, inhibiting a more complete recovery. While the first breakdown is induced by the particular setup of the initial state, the subsequent oscillations can be seen, more appropriately, as fluctuations around an equilibrium state toward which the system is approaching.

4. Concluding remarks

The nature of vacillations and their underlying mechanisms in SW models of the stratosphere were reexamined using two PV-based numerical models in a PV, velocity divergence, acceleration divergence representation:
the recently developed “diabatic contour-advective semi-Lagrangian” algorithm, called DCASL, and (2) a standard semi-Lagrangian scheme for PV, called SL. The DCASL algorithm offers high Lagrangian resolution. Extensive numerical experiments and new diagnostics were specifically designed to (a) examine the possible mechanisms proposed for the vacillation cycles and (b) remove the uncertainties arising from the use of

Fig. 8. As in Fig. 5, but for the experiment starting from a state of rest with ramped topography.

Fig. 9. Time evolution of (a) the total PV gradient, (b) the mean PV gradient, and (c) the length of the $Q = 2.2 \times 10^{-9} \text{ m}^{-1} \text{ s}^{-1}$ contour in the experiment starting from a state of rest. The results are for the DCASL algorithm with $256 \times 256$ resolution.
hyperdiffusion acting on the vorticity field in a vorticity–divergence–mass representation of the SW model. To focus on the mechanisms responsible for vacillations, the presentation was limited to strong topographic forcing with a fixed representative value of 10 days for the radiative relaxation time. In addition to the experiment of RW04 that starts from a balanced zonal jet characteristic of the midwinter middle stratosphere, called the

![Graphs and images]

FIG. 10. As in Fig. 5, but for the experiment that starts from $t = t_s = 100$ days of the standard experiment using diabatic PV inversion.

FIG. 11. As in Fig. 9, but for the experiment that starts from $t = t_s = 100$ days of the standard experiment using diabatic PV inversion.
standard experiment, two other sets of experiments were carried out: the first starting from a state of rest, and the second starting from a rebalanced fully developed state. These new sets of experiments allowed us to probe the extremes of complexity in the initial PV distribution.

For both the standard experiment and the one starting from a state of rest, vacillations appear with a large amplitude and then decay rapidly. This is particularly true as the effective resolution increases in DCASL. For the experiment starting from a fully developed, properly initialized state using diabatic PV inversion, the vacillations virtually cease to exist after about two to three multiples of the dominant period of the first vacillation in the standard experiment. In this case, over the longer time evolution, the vacillations may appear randomly with significantly less amplitude. The three sets of experiments together point to the absence of robust vacillations. The two mechanisms proposed by RW04—the out-of-phase action of the diabatic and dissipative fluxes and systematic variations in PV gradients and Rossby waves—do not appear capable of generating coherent variability in the form of regular vacillations. Instead of out-of-phase action, it is in-phase action that is dominant in most cases, prohibiting vacillations of note to take place. The simultaneous action of the wave forcing, radiative relaxation, and dissipation can be understood as inherent fluctuations superimposed on a damped, nonlinear oscillator.

The existence of coherent modes of internal variability for the stratosphere (Scott and Polvani 2006), their possible impact on the tropospheric circulation (Baldwin and Dunkerton 2001), and their evolution over time scales longer than those considered in medium-range weather forecasting is of prime importance in reducing one of the major uncertainties in the climate system. With direct inclusion of PV as a prognostic variable, the PV-based SW algorithms used in this study lead us to a more complex picture of variability, with irregularity present even at resolutions comparable to T42 in a spectral transform algorithm. In light of our results, the transition from regularity to irregularity with increasing (horizontal) resolution in the spectral transform results

**Fig. 12.** The autocorrelation for (a)–(c) the mass within, (d)–(f) the dissipative mass flux across, and (g)–(i) the diabatic mass flux across the edge of the polar vortex represented by $Q = 2.2 \times 10^{-3} \text{ m}^{-1} \text{ s}^{-1}$. (j)–(l) The cross correlation between the dissipative and diabatic mass fluxes. Results are for (top) the standard experiment, (middle) the one starting from a state of rest, and (bottom) the one starting from the fully developed state at $t = t_s = 100$ days.
of RW04 and Scott and Polvani (2006) can be traced to the improvement in the representation of PV with increasing resolution. Lower-resolution general circulation and climate models can thus be expected to show enhanced variability if they are reformulated in terms of PV. PV-based algorithms have been developed for the multilevel isentropic-coordinate primitive equations on the sphere, providing an opportunity to address the question of internal variability in a more realistic setting. This appears to be an essential next step, especially as the SW model likely misrepresents the transition between strong and weak vortex states, which is a baroclinic process in the real stratosphere (see, e.g., Castanheira et al. 2009; Matthewman et al. 2009). Further, compared to PV dissipation in the SW model, vertical shear and vertical-scale-selective radiative damping (Haynes and Ward 1993) may provide a different way of removing small-scale PV structures, with the potential to alter the phase relation between the diabatic and dissipative fluxes critical to the vacillations.

![Figure 13](image13.png)

**Fig. 13.** The time evolution of the mass within the edge of the polar vortex for the standard experiment but with $H_b = (3000, 4000, 5000, 6000)$ m.

![Figure 14](image14.png)

**Fig. 14.** As in Fig. 4, but for $H_b = 4000$ m.
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