The mathematical derivation of ELM

by

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November 28, 2002

Contents

Introduction

This document provides details of the mathematical derivation of the *Quasi-Geostrophic Ellipsoidal Model* (ELM). It intends to log the mathematical basis of the formulation (including full demonstrations where available).

The document is organised as follows:

- Chapter 1 recalls the QG equations with constant Coriolis and buoyancy f and N, as assumed in the present model.
- Chapter 2 presents the modelling of the external streamfunction induced by a PV ellipsoid using point vortices (singularities).
- Chapter 3 gives the hamiltonian formulation for the equations of motion for interacting ellipsoids.
- Chapter 4 explains the numerical procedure used to obtain steady states for two interacting ellipsoids.

This 'beta' version of the document aims only at overviewing the full, detailed equations, as used in the codes. It does not aim at being fully pedagogic, at least in its present form. I wish this document to be 'alive' and to be improved from readers comments.

At last, I hope you will find the current version helpful.

Jean N. Reinaud

Chapter 1

The quasi-geostrophic model

The quasi-geostrophic (QG) equations are obtained from an asymptotic expansion of Euler's equations for $\epsilon = H/L \ll 1$, where H and L are the characteristic vertical and horizontal length scales, and for $F_r^2 << R_o << 1$ where F_r and R_o are respectively the Froude and Rossby numbers.

Using incompressibility and the fact that advection is constrained to be layerwise two-dimensional,

$$\mathbf{u} = -\boldsymbol{\nabla} \times \psi \hat{\mathbf{e}}_3 \Leftrightarrow \mathbf{u} = \mathcal{L} \boldsymbol{\nabla} \psi \tag{1.1}$$

with

$$\mathcal{L} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2}$$

In an adiabatic, inviscid fluid, the potential vorticity (PV) is materially conserved

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{1.3}$$

Finally, in the QG model, the streamfunction can be deduced from the PV distribution by a linear inversion relation:

$$\Delta \psi = q. \tag{1.4}$$

In this last equation, the vertical coordinate has been stretched by the ratio N/f resulting in an isotropic relation between PV distribution and streamfunction.

Chapter 2

Modelling the ellipsoid

2.1 Exact solution

The external streamfunction ψ_{out} induced by a homogeneous ellipsoid of PV in standard position whose axis half-lengths are a, b and c is given by (see Chandrasekhar, 1969)

$$\psi_{out}(x,y,z) = -\frac{3\kappa}{4}\int_{\lambda}^{\infty}\frac{1}{\sqrt{(u+a^2)(u+b^2)(u+c^2)}}\left(1-\frac{x^2}{u+a^2}-\frac{y^2}{u+b^2}-\frac{z^2}{u+c^2}\right)\mathrm{d}u \qquad (2.1)$$

where λ is the largest root of the cubic equation

$$\frac{x^2}{\lambda+a^2}+\frac{y^2}{\lambda+b^2}+\frac{z^2}{\lambda+c^2}=1 \qquad (2.2)$$

and $\kappa = qV/(4\pi) = qabc/3$.

Denoting

$$f = a^2+\lambda, \qquad (2.3)$$

$$g = b^2+\lambda, \qquad (2.4)$$

$$h = c^2+\lambda, \qquad (2.5)$$

$$R_F(x,y,z) = \frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}u}{\sqrt{(t+x)(t+y)(t+z)}}, \qquad (2.6)$$

$$R_D(x,y,z) = \frac{3}{2}\int_0^{\infty}\frac{\mathrm{d}u}{\sqrt{(t+x)(t+y)(t+z)^3}}, \qquad (2.7)$$

the streamfunction can be usefully rewritten as

$$\psi_{out} = -\frac{3\kappa}{2}\left(R_F(f,g,h)-\frac{1}{3}\left(x^2R_D(g,h,f)+y^2R_D(h,f,g)+z^2R_D(f,g,h)\right)\right) \qquad (2.8)$$

where R_F and R_D are respectively the standard elliptic integrals of the first and second kind. This exact formula has been used to test numerically the accuracy of the proposed discrete models detailed below.

2.2 Taylor expansion of the streamfunction

The streamfunction induced by a homogeneous volume V of PV is given by the following Green's integral:

$$\psi(\mathbf{x}) = -\frac{q}{4\pi} \iiint_V \frac{\mathrm{d}V'}{|\mathbf{x} - \mathbf{x}'|}. \tag{2.9}$$

The integrand can be expanded using Legendre's polynomials P_n:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r\sqrt{1 - 2\frac{\mathbf{x}\cdot\mathbf{x}'}{rr'}\frac{r'}{r} + \frac{r'^2}{r^2}}} = \sum_{n=0}^{\infty} \frac{1}{r^{2n+1}} P_n\left(\frac{\mathbf{x}\cdot\mathbf{x}'}{rr'}\right)(rr')^n \tag{2.10}$$

where $r = |\mathbf{x}|$ and $r' = |\mathbf{x}'|$.

In our case, the volume of PV is an ellipsoid in standard position, centred at the origin and with $c \geq b \geq a$. (These choices do not alter the generality of the result since Poisson's equation is mathematically isotropic: we can always go back to this situation, in a relevant reference frame.)

We expand the streamfunction to the 6^{th} order in $1/r$. Considering the symmetry properties of an ellipsoid, only terms in odd power in $1/r$ are non-zero. The three non-zero terms are:

$$\psi = \psi_0 + \psi_1 + \psi_2 + \mathcal{O}(r^{-7}) \tag{2.11}$$

where

$$\psi_0 = -\frac{\kappa}{r}, \tag{2.12}$$

$$\psi_1 = \frac{q}{8\pi r^5}\left(r^2 \iiint \mathbf{x}'^2 \mathrm{d}V' - 3\iiint (\mathbf{x}\cdot\mathbf{x}')^2 \mathrm{d}V'\right), \tag{2.13}$$

and

$$\psi_2 = -\frac{q}{32\pi r^9}\left(3r^4 \iiint \mathbf{x}'^4 \mathrm{d}V' - 30r^2 \iiint (\mathbf{x}\cdot\mathbf{x}')^2\mathbf{x}'^2 \mathrm{d}V' + 35\iiint (\mathbf{x}\cdot\mathbf{x}')^4 \mathrm{d}V'\right). \tag{2.14}$$

We next define from the previous integrals 9 'position' coefficients:

$$M_x^{(2)} = 3x^2 - r^2, \tag{2.15}$$
$$M_y^{(2)} = 3y^2 - r^2, \tag{2.16}$$
$$M_z^{(2)} = 3z^2 - r^2, \tag{2.17}$$
$$M_x^{(4)} = 3r^4 - 30x^2r^2 + 35x^4, \tag{2.18}$$
$$M_y^{(4)} = 3r^4 - 30y^2r^2 + 35y^4, \tag{2.19}$$
$$M_z^{(4)} = 3r^4 - 30z^2r^2 + 35z^4, \tag{2.20}$$
$$M_{x,y}^{(4)} = r^4 - 5(x^2 + y^2)r^2 + 35x^2y^2, \tag{2.21}$$
$$M_{x,z}^{(4)} = r^4 - 5(x^2 + z^2)r^2 + 35x^2z^2, \tag{2.22}$$
$$M_{y,z}^{(4)} = r^4 - 5(y^2 + z^2)r^2 + 35y^2z^2. \tag{2.23}$$

which only depend on the location of the point at which we evaluate the streamfunction. Note that the $M^{(2)}$ coefficients are of order r^2 and the $M^{(4)}$ coefficients are of order r^4.

Only 5 of these 9 coefficients are independant since we have the 4 following algebraic relations:

$$M_x^{(2)} + M_y^{(2)} + M_z^{(2)} = 0, \tag{2.24}$$
$$M_x^{(4)} + M_y^{(4)} + 2M_{x,y}^{(4)} = M_z^{(4)}, \tag{2.25}$$
$$M_x^{(4)} + M_z^{(4)} + 2M_{x,z}^{(4)} = M_y^{(4)}, \tag{2.26}$$
$$M_y^{(4)} + M_z^{(4)} + 2M_{y,z}^{(4)} = M_x^{(4)}. \tag{2.27}$$

Introducing

$$\eta = \sqrt{b^2 - a^2}, \tag{2.28}$$
$$\tau = \sqrt{c^2 - a^2}, \tag{2.29}$$

which are the focal lengths for the ellipses lying in the $x - y$ and $x - z$ planes, we may compute the following 5 even, independent, moments for the ellipsoids as follows

$$I_y^{(2)} = \frac{1}{4\pi\kappa} \iiint_{\mathcal{E}} q\left(y'^2 - x'^2\right) \mathrm{d}V' = \frac{1}{5}\eta^2, \tag{2.30}$$
$$I_z^{(2)} = \frac{1}{4\pi\kappa} \iiint_{\mathcal{E}} q\left(z'^2 - x'^2\right) \mathrm{d}V' = \frac{1}{5}\tau^2, \tag{2.31}$$
$$I_x^{(4)} = \frac{1}{4\pi\kappa} \iiint_{\mathcal{E}} q\left(x'^4 + 3y'^2z'^2 - 3x'^2(y'^2 + z'^2)\right) \mathrm{d}V' = \frac{3}{35}\eta^2\tau^2, \tag{2.32}$$
$$I_y^{(4)} = \frac{1}{4\pi\kappa} \iiint_{\mathcal{E}} q\left(y'^4 + 3x'^2z'^2 - 3y'^2(x'^2 + z'^2)\right) \mathrm{d}V' = \frac{3}{35}\eta^2(\eta^2 - \tau^2), \tag{2.33}$$
$$I_z^{(4)} = \frac{1}{4\pi\kappa} \iiint_{\mathcal{E}} q\left(z'^4 + 3y'^2z'^2 - 3z'^2(x'^2 + y'^2)\right) \mathrm{d}V' = \frac{3}{35}\tau^2(\tau^2 - \eta^2) \tag{2.34}$$

in terms of which the streamfunction can now be written as

$$-\frac{\psi}{\kappa} \equiv \hat{\psi} = \hat{\psi}_0 + \hat{\psi}_1 + \hat{\psi}_2 + \mathcal{O}(r^7), \tag{2.35}$$

$$\hat{\psi}_0 = \frac{1}{r}, \tag{2.36}$$

$$\hat{\psi}_1 = \frac{1}{2r^5}\left(M_y^{(2)} I_y^{(2)} + M_z^{(2)} I_z^{(2)}\right), \tag{2.37}$$

$$\hat{\psi}_2 = \frac{1}{8r^9}\left(M_x^{(4)} I_x^{(4)} + M_y^{(4)} I_y^{(4)} + M_z^{(4)} I_z^{(4)}\right). \tag{2.38}$$

Then, a distribution of PV sharing the same moments $I^{(n)}$ will induce the same external flow (at least to the order considered) as the ellipsoid.

2.3 An elliptic sheet of PV

We conjecture that a singular elliptical sheet distribution of PV γ_s placed in the $y - z$ plane induces the same external streamfunction as the one from the homogeneous PV ellipsoid. The half-lengths of the ellipse are respectively η and τ and the PV is assumed to be radially distributed (in elliptical coordinates) according to the law γ_s:

$$\rho \equiv \sqrt{\frac{y^2}{\eta^2} + \frac{z^2}{\tau^2}}, \tag{2.39}$$

$$y = \eta\rho\cos\theta, \quad z = \tau\rho\sin\theta, \tag{2.40}$$

$$\gamma_s(x, \rho) = \delta(x)\Gamma(\rho), \tag{2.41}$$

$$\Gamma(\rho) = \alpha\left(1 - \rho^2\right)^p, \tag{2.42}$$

where δ is the Dirac distribution. We are going to demonstrate that this singular planar structure can share the same first moments $I^{(n)}$ as the ones of the ellipsoid, and consequently induce (at the order considered at least) the same external streamfunction.

We first determine the coefficients α and p by enforcing that both the total circulation of the sheet and the coefficient $I_y^{(2)}$ are the same as the ones for the ellipsoid.

Using $\mathrm{d}y\mathrm{d}z = \eta\tau\rho\mathrm{d}\rho\mathrm{d}\theta$, the circulation is

$$4\pi\kappa = \eta\tau \int_0^{2\pi} \int_0^1 \Gamma(\rho)\rho\,\mathrm{d}\rho\mathrm{d}\theta, \tag{2.43}$$

$$= \pi\eta\tau\alpha \int_0^1 (1-u)^p \mathrm{d}u = \frac{\pi\eta\tau\alpha}{p+1} \tag{2.44}$$

giving

$$\alpha = \frac{4\kappa}{\eta\tau}(p+1). \tag{2.45}$$

Matching next the second moment in y, we have

$$I_y^{(2)} = \frac{\eta\tau}{4\pi\kappa}\int_1^1\int_0^{2\pi} y^2\Gamma(\rho)\rho \mathrm{d}\rho\mathrm{d}\theta, \tag{2.46}$$

$$= \frac{\eta^3\tau}{4\pi\kappa}\int_0^1\int_0^{2\pi} \rho^3\alpha\left(1-\rho^2\right)^p\cos^2\theta\mathrm{d}\theta\mathrm{d}\rho, \tag{2.47}$$

$$= \frac{\eta^2(p+1)}{2}\int_0^1(1-u)^p u\mathrm{d}u = \frac{\eta^2}{2(p+2)} \tag{2.48}$$

using (2.45). By (2.30), we must have $I_y^2 = \frac{1}{5}\eta^2$, and therefore

$$p = \frac{1}{2}, \tag{2.49}$$

$$\alpha = \frac{6\kappa}{\eta\tau}. \tag{2.50}$$

$$\tag{2.51}$$

We can similarly verify that the second moment $I_z^{(2)}$ is matched as well:

$$I_z^{(2)} = \frac{\tau^2}{2(p+2)} = \frac{1}{5}\tau^2. \tag{2.52}$$

This guarantees that the streamfunction induced by the elliptical sheet is equal to the streamfunction of the ellipsoid at least to the 4^{th} order in $1/r$. Moreover, we can verify that the fourth order moments are matched as well:

$$I_x^{(4)} = \frac{\eta\tau\alpha}{4\pi\kappa}\int_0^1\int_0^{2\pi} 3y'^2z'^2\sqrt{1-\rho^2}\rho\mathrm{d}\rho\mathrm{d}\theta, \tag{2.53}$$

$$= \frac{9\eta^2\tau^2}{2\pi}\int_0^1\frac{1}{2}u^2\sqrt{1-u}\mathrm{d}u\int_0^{2\pi}\frac{1}{4}\sin^2(2\theta)\mathrm{d}\theta, \tag{2.54}$$

with

$$\frac{1}{2}\int_0^1 u^2\sqrt{1-u}\mathrm{d}u = \frac{8}{105}, \tag{2.55}$$

$$\frac{1}{4}\int_0^{2\pi}\sin^2(2\theta)\mathrm{d}\theta = \frac{\pi}{4}, \tag{2.56}$$

leading to

$$I_x^{(4)} = \frac{3}{35}\eta^2\tau^2. \tag{2.57}$$

And, similarly

$$I_y^{(4)} = \frac{\alpha\eta\tau}{4\pi\kappa}\int_0^1\int_0^{2\pi} y'^4\sqrt{1-\rho^2}\rho\mathrm{d}\rho\mathrm{d}\theta - I_x^{(4)}, \tag{2.58}$$

$$= \frac{3\eta^4}{2\pi}\int_0^1 \frac{1}{2}u^2\sqrt{1-u}\mathrm{d}u\int_0^{2\pi}\cos^4(\theta)\mathrm{d}\theta - I_x^{(4)}, \tag{2.59}$$

while

$$\int_0^{2\pi}\cos^4(\theta)\mathrm{d}\theta = \frac{3}{4}\pi, \tag{2.60}$$

leading to

$$I_y^{(4)} = \frac{3}{35}(\eta^4 - \eta^2\tau^2). \tag{2.61}$$

Finally,

$$I_z^{(4)} = \frac{\alpha\eta\tau}{4\pi\kappa}\int_0^1\int_0^{2\pi} z'^4\sqrt{1-\rho^2}\rho\mathrm{d}\rho\mathrm{d}\theta - I_x^{(4)}, \tag{2.62}$$

$$I_y^{(4)} = \frac{3\tau^4}{2\pi}\int_0^1 \frac{1}{2}u^2\sqrt{1-u}\mathrm{d}u\int_0^{2\pi}\sin^4(\theta)\mathrm{d}\theta - I_x^{(4)}, \tag{2.63}$$

while

$$\int_0^{2\pi}\sin^4(\theta)\mathrm{d}\theta = \frac{3}{4}\pi, \tag{2.64}$$

leading to

$$I_y^{(4)} = \frac{3}{35}(\tau^4 - \eta^2\tau^2). \tag{2.65}$$

This means that the elliptical sheet of PV is relevant to the 6^{th} order at least. We conjecture that this sheet of PV is relevant at all orders, or in other words, it induces the same external streamfunction as our ellipsoid of uniform PV.

Ellipsoid moments	Elliptical sheet momments
$I^{(0)}$	$\mu_{0,0}/\kappa$
$I_y^{(2)}$	$\mu_{1,0}/\kappa$
$I_z^{(2)}$	$\mu_{0,1}/\kappa$
$I_x^{(4)}$	$\mu_{1,1}/\kappa$
$I_y^{(4)}$	$(\mu_{2,0} - \mu_{1,1})/\kappa$
$I_z^{(4)}$	$(\mu_{0,2} - \mu_{1,1})/\kappa$

Table 2.1: Correspondance between the ellipsoid moments and the elliptical sheet moments.

2.4 Modelling the elliptic sheet

We next define sheet moments $\mu_{m,n}$ related to the moments $I^{(k)}$ introduced before.

$$\mu_{m,n} = \frac{1}{4\pi} \iint_{\rho<1} \Gamma(\rho) y^{2m} z^{2n} \mathrm{d}y \mathrm{d}z, \tag{2.66}$$

$$\mu_{m,n} = \kappa M_{m+n} \lambda_{m,n}, \tag{2.67}$$

where

$$M_k = 6 \int_0^1 \rho^{2k+1} \sqrt{1-\rho^2} \mathrm{d}\rho, \tag{2.68}$$

$$\lambda_{m,n} = \eta^{2m} \tau^{2n} \int_0^{2\pi} \cos^{2m}\theta \sin^{2n}\theta \mathrm{d}\theta. \tag{2.69}$$

The correspondance between the $I^{(p)}$ moments for the ellipsoid and the $\mu_{m,n}$ for the ellipstic sheet are given in table 2.1 up to $p = 4$.

Up to $m + n = 3$ we have

$$M_0 = 2 \tag{2.70}$$

$$M_1 = \frac{4}{5} \tag{2.71}$$

$$M_2 = \frac{16}{35} \tag{2.72}$$

$$M_3 = \frac{32}{105} \tag{2.73}$$

and

$$\lambda_{0,0} = \frac{1}{2} \tag{2.74}$$

$$\lambda_{1,0} = \frac{\eta^2}{4} \quad \text{and} \quad \lambda_{0,1} = \frac{\tau^2}{4} \tag{2.75}$$

$$\lambda_{2,0} = \frac{3\eta^4}{16} \quad , \quad \lambda_{1,1} = \frac{\eta^2\tau^2}{16} \quad \text{and} \quad \lambda_{0,2} = \frac{3\tau^4}{16} \tag{2.76}$$

$$\lambda_{3,0} = \frac{5\eta^6}{32} \quad , \quad \lambda_{2,1} = \frac{\eta^4\tau^2}{32} \quad , \quad \lambda_{1,2} = \frac{\eta^2\tau^4}{32} \quad \text{and} \quad \lambda_{0,3} = \frac{5\tau^6}{32} \quad , \tag{2.77}$$

To allow a fast means to evaluate the external streamfunction, we approximate the elliptic sheet by a simple distribution of a few point vortices. Because of the two-fold symmetry properties of the elliptic sheet (in y and z), the point vortex distribution must be symmetric as well. Then we have to ensure that the combinations of vortices match the moments $\mu_{i,j}$ up to a certain number to reach the order of accuracy wanted.

We use four basic combinations of point vortices combinations:

- **A central vortex** which provides one degree of freedom while ensuring the symmetry properties: its strength κ_0. Note that because of its central position, this vortex only contributes to the moment $\mu_{0,0}$
- **A vortex pair** which provides 2 degrees of freedom : the (equal) strength of each vortex κ_p and their location. Two choices are possible as the vortices can lie along either the y or the z axis. Note that the pair contributes to moments only in one direction i.e. $\mu_{i,0}$ or $\mu_{0,i}$.
- **A quartet of vortices** which provides three degrees of freedom: the (equal) strength of its four vortices κ_q and their location in the (y, z)-plane: y_q and z_q. Note that two configurations preserve the required two-fold symmetry: quartet A in which the 4 point vortices are located at $(y_q, 0)$, $(-y_q, 0)$, $(0, z_q)$, $(0, -z_q)$ or quartet B in which the 4 point vortices are located at (y_q, z_q), $(-y_q, z_q)$, $(y_q, -z_q)$, $(-y_q, -z_q)$. Note also that the quartets contribute to moments in both directions ($\mu_{i,0}$ and $\mu_{0,j}$) but only the quartet B contributes to cross-moments (e.g. $\mu_{i,j}$ with i and $j \neq 0$).
- **A sextet of vortices** which provides four degrees of freedom. The sextet can be seen as a quartet B (3 degrees) plus a pair of vortices of same strength (providing an additional degree from the position of the vortices along the y or z axis). There are again two possible configurations, depending on location of the pair. Note that the sextet can contribute to all moments $\mu_{i,j}$. When the pair of vortices lies along z (longest axis), it will be referred to as sextet A, otherwise it will be referred to as sextet B.

We now present the different models, depending on the required accuracy for the streamfunction:

Order $(1/r)$:

At this order of accuracy, we only need to match the moment $\mu_{0,0} = \kappa$. A single vortex of strength $\kappa_0 = \kappa$ placed at the origin is sufficient.

Order $(1/r^3)$:

We now need to match 3 moments: $\mu_{0,0} = \kappa$, $\mu_{1,0} = \kappa\eta^2/5$ and $\mu_{0,1} = \kappa\tau^2/5$. A quartet of vortices, which provides three degrees of freedom, is a possible solution. Note that a pair and a central vortex would also provide 3 degrees of freedom but they can only contribute to moments in one direction (the direction of the axis on which the pair is lying), and therefore cannot be a solution to our problem.

For quartet A the equations are

$$4\kappa_q = \kappa, \tag{2.78}$$
$$2\kappa_q y_q^2 = \kappa\eta^2/5, \tag{2.79}$$
$$2\kappa_q z_q^2 = \kappa\tau^2/5, \tag{2.80}$$

which leads to

$$\kappa_q = \frac{\kappa}{4}, \tag{2.81}$$
$$y_q = \sqrt{\frac{2}{5}}\eta, \tag{2.82}$$
$$z_q = \sqrt{\frac{2}{5}}\tau. \tag{2.83}$$

Note that the vortices lie on an elliptic ring of radius $\rho = \sqrt{2/5}$.

For quartet B the equations are

$$4\kappa_q = \kappa, \tag{2.84}$$
$$4\kappa_q y_p^2 = \kappa\eta^2/5, \tag{2.85}$$
$$4\kappa_q z_p^2 = \kappa\tau^2/5. \tag{2.86}$$

which leads to

$$\kappa_q = \frac{\kappa}{4}, \tag{2.87}$$
$$y_q = \sqrt{\frac{1}{5}}\eta, \tag{2.88}$$
$$z_q = \sqrt{\frac{1}{5}}\tau. \tag{2.89}$$

Note that the vortices lie again along the ring of radius $\rho = \sqrt{2/5}$ but are now located at the elliptic angles $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$.

Numerical tests comparing the streamfunction obtained from these two models with the streamfunction using the exact solution (2.8) show that quartet B is slightly more accurate than quartet A, see figure 2.1.

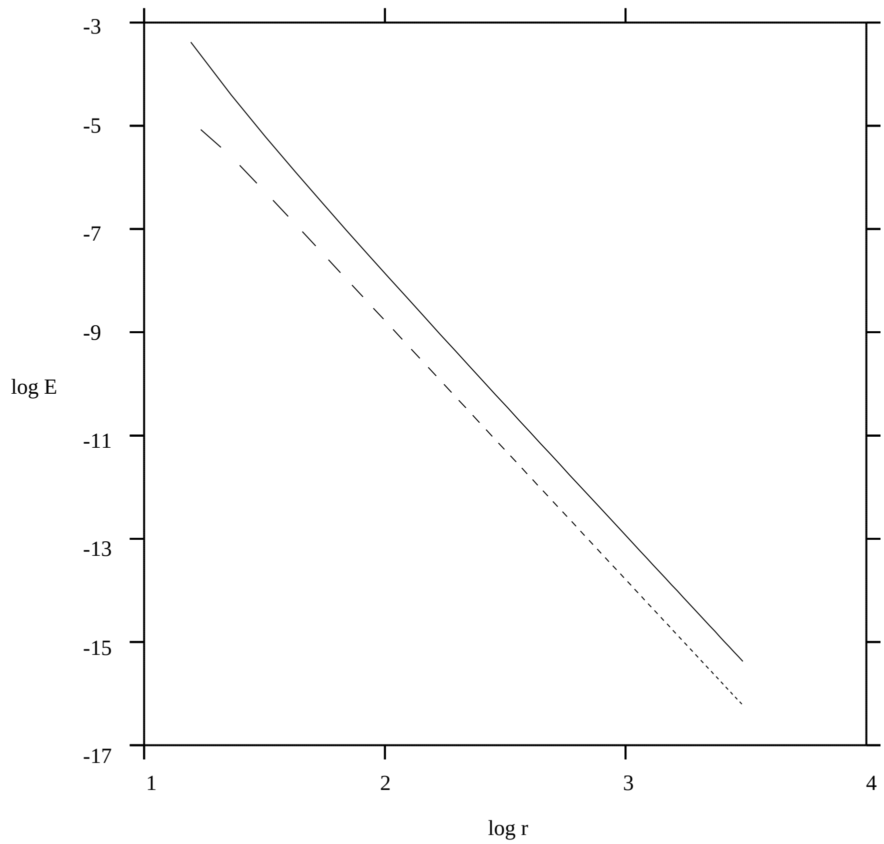

Figure 2.1: Illustration of the error (in log/log scales) for quartet A (solid line) and quartet B (dashed line). The error is computed along the x axis for an ellipsoid with $a = 1$, $b = 2$, and $c = 3$. The PV is set to 1.

Order $1/r^5$

We now need to match 6 moments: $\mu_{0,0} = \kappa$, $\mu_{1,0} = \kappa\eta^2/5$, $\mu_{0,1} = \kappa\tau^2/5$, $\mu_{2,0} = 3\kappa\eta^4/35$, $\mu_{1,1} = \kappa\eta^2\tau^2/35$ and $\mu_{0,2} = 3\kappa\tau^4/35$. Three pairs of vortices provide 6 degrees of freedom. However, they cannot be used here as they do not create any $\mu_{1,1}$ moment. Therefore we choose a combination of 7 vortices: a central vortex, a quartet B and a pair lying along the z axis. This provides 6 degrees of freedom as well. The equations are :

$$\kappa_0 + 2\kappa_p + 4\kappa_q = \kappa, \tag{2.90}$$
$$4\kappa_q y_q^2 = \kappa\eta^2/5, \tag{2.91}$$
$$2\kappa_p z_p^2 + 4\kappa_q z_q^2 = \kappa\tau^2/5, \tag{2.92}$$
$$4\kappa_q y_q^4 = 3\kappa\eta^4/35, \tag{2.93}$$
$$2\kappa_p z_p^2 + 4\kappa_q z_q^4 = 3\kappa\tau^4/35, \tag{2.94}$$
$$4\kappa_q y_q^2 z_q^2 = \kappa\eta^2\tau^2/35. \tag{2.95}$$

Dividing (2.93) and (2.95) by (2.91) we obtain

$$y_q = \sqrt{\frac{3}{7}}\eta, \tag{2.96}$$
$$z_q = \sqrt{\frac{1}{7}}\tau. \tag{2.97}$$

Then, using (2.91) we obtain

$$\kappa_q = \frac{7\kappa}{60}. \tag{2.98}$$

Then equations (2.92) and (2.94) give respectively

$$\kappa_p z_p^2 = \kappa\tau^2/15, \tag{2.99}$$
$$\kappa_p z_p^4 = 4\kappa\tau^4/105. \tag{2.100}$$

Then

$$z_p = \sqrt{\frac{4}{7}}\tau, \tag{2.101}$$
$$\kappa_p = \frac{7}{60}\kappa \quad (= \kappa_q)\,. \tag{2.102}$$

It appears that the quartet plus the pair combine into a sextet (sextet A) lying along an elliptic ring of radius $\rho = \sqrt{4/7}$. Moreover neighbouring singularities are separated uniformly by an elliptic angle of $\pi/3$.

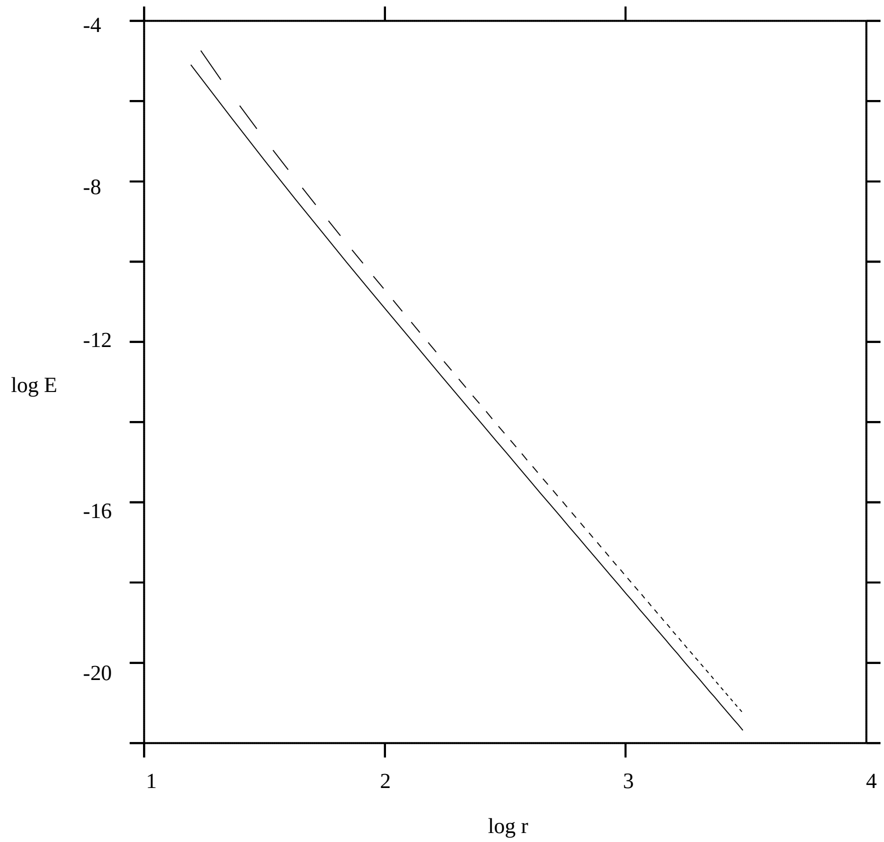

Figure 2.2: Illustration of the error (in log/log scales) for sextet A (solid line) and sextet B (dashed line). The error is computed along the x axis for an ellipsoid with $a = 1$, $b = 2$, and $c = 3$. The PV is set to 1.

Finally we have from equation (2.91),

$$\kappa_0 = \frac{3\kappa}{10}. \tag{2.103}$$

Using a pair along the y-axis instead yields the same results, rotated by $\pi/6$ (sextet B). However, this second configuration has been shown to be slightly less accurate, see figure 2.2 .

Order $1/r^7$

We now need to match 10 moments: $\mu_{0,0} = \kappa$, $\mu_{1,0} = \kappa\eta^2/5$, $\mu_{0,1} = \kappa\tau^2/5$, $\mu_{2,0} = 3\kappa\eta^4/35$, $\mu_{1,1} = \kappa\eta^2\tau^2/35$, $\mu_{0,2} = 3\kappa\tau^4/35$, $\mu_{3,0} = 5\kappa\eta^6/105$, $\mu_{2,1} = \kappa\eta^4\tau^2/105$, $\mu_{1,2} = \kappa\eta^2\tau^4/105$ and $\mu_{0,3} = 5\kappa\tau^6/105$.

The algebra necessary to solve the problem becomes much more difficult. Then, following the previous results, we anticipate a solution that uses 2 sextets, s_1 and s_2 (each sextet lying on a ring and the vortices being spaced by an angle of $\pi/3$) and a central vortex, i.e. 13 vortices overall. We have tried to use other distributions, using less singularities, but these distributions failed to match all the required moments.

We denote u the squared radius of the ring along which the sextet s_1 lies and v for s_2. We place the

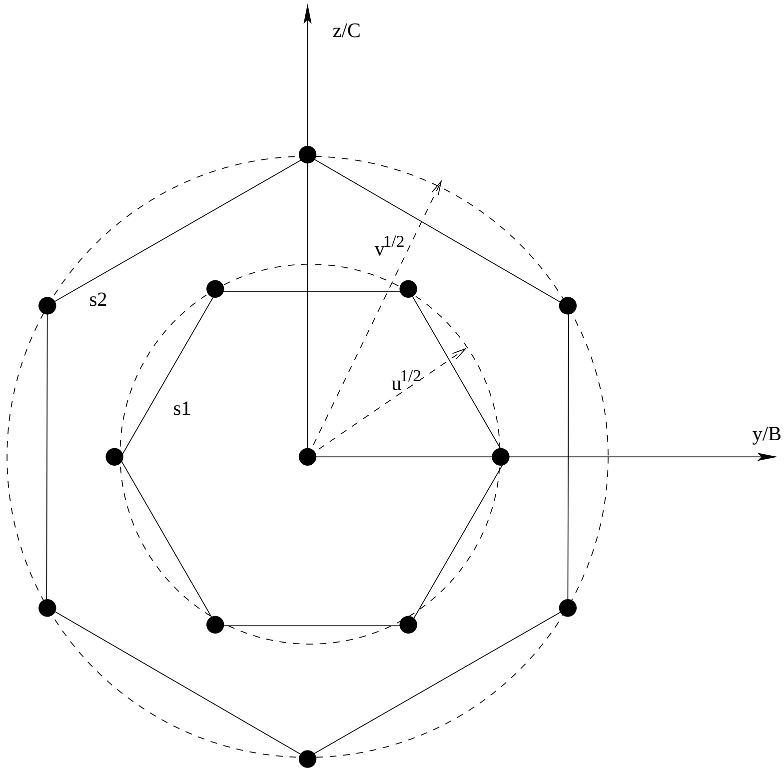

Figure 2.3: Sketch of the of the model with two sextets and a central vortex.

vortices of the sextet s_1 at elliptic angles of 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$ and $5\pi/3$ (sextet B) and at $\pi/6$, $\pi/2$, $5\pi/6$, $7\pi/6$, $9\pi/6$ and $11\pi/6$ for s_2 (sextet A), see fig. 2.3 for a sketch of the geometry.

We determine the 5 unknown variables (the three strengths κ_0, κ_{s_1} and κ_{s_2} and the two radii $\sqrt{u}$ and $\sqrt{v}$) describing the combination and verify that the model can match the 10 required moments.

The 10 equations are

$$\kappa_0 + 6(\kappa_{s_1} + \kappa_{s_2}) = \kappa, \tag{2.104}$$

$$\eta^2(\kappa_{s_1} 3u + \kappa_{s_2} 3v) = \kappa\eta^2/5, \tag{2.105}$$

$$\tau^2(\kappa_{s_1} 3u + \kappa_{s_2} 3v) = \kappa\tau^2/5, \tag{2.106}$$

$$\eta^4(\kappa_{s_1}(36u^2/16) + \kappa_{s_2}(36v^2/16)) = 3\kappa\eta^4/35, \tag{2.107}$$

$$\tau^4(\kappa_{s_2}(36u^2/16) + \kappa_{s_2}(36v^2/16)) = 3\kappa\tau^4/35, \tag{2.108}$$

$$\eta^2\tau^2(\kappa_{s_1}(12u^2/16) + \kappa_{s_1}(12v^3/16)) = \kappa\eta^2\tau^2/35, \tag{2.109}$$

$$\eta^6(\kappa_{s_1}(33u^3/64) + 4\kappa_{s_2}(27v^3/64)) = \kappa\eta^6/21, \tag{2.110}$$

$$\tau^6(4\kappa_{s_1}(27u^3/64) + 4\kappa_{s_2}(33v^3/64)) = \kappa\tau^6/21, \tag{2.111}$$

$$\eta^4\tau^2(\kappa_{s_1}(3u^3/16) + kappa_{s_2}(9v^3/16)) = \kappa\eta^4\tau^2/105, \tag{2.112}$$

$$\eta^2\tau^4(\kappa_{s_1}(9u^3/16) + \kappa_{s_2}(3v^3/16)) = \kappa\eta^2\tau^4/105. \tag{2.113}$$

As equations (2.105) and (2.106) and also (2.107), (2.108) and (2.109) are combinations of the same equation, we can reduce the system to:

$$\kappa_0 + 6(\kappa_{s1} + \kappa_{s_2}) = \kappa, \tag{2.114}$$
$$\kappa_{s1} u + \kappa_{s_2} v = \kappa/15, \tag{2.115}$$
$$\kappa_{s1} u^2 + \kappa_{s_2} v^2 = 4\kappa/105, \tag{2.116}$$
$$33\kappa_{s1} u^3 + 27\kappa_{s_2} v^3 = 16\kappa/21, \tag{2.117}$$
$$27\kappa_{s1} u^3 + 33\kappa_{s_2} v^3 = 16\kappa/21, \tag{2.118}$$
$$9\kappa_{s1} u^3 + 3\kappa_{s_2} v^3 = 16\kappa/105, \tag{2.119}$$
$$3\kappa_{s1} u^3 + 9\kappa_{s_2} v^3 = 16\kappa/105. \tag{2.120}$$

Equations (2.117) and (2.118) imply that $\kappa_{s_1} u^3 = \kappa_{s_2} v^3 = 4\kappa/315$. Consequently, equations (2.119) and (2.120) are automatically satisfied.

The system reduces to 5 independent nonlinear equations:

$$\kappa_0 + 6(\kappa_{s_1} + \kappa_{s_2}) = \kappa, \tag{2.121}$$
$$\kappa_{s_1} u + \kappa_{s_2} v = \kappa/15, \tag{2.122}$$
$$\kappa_{s_1} u^2 + \kappa_{s_2} v^2 = 4\kappa/105, \tag{2.123}$$
$$\kappa_{s_1} u^3 = 4\kappa/315, \tag{2.124}$$
$$\kappa_{s_2} v^3 = 4\kappa/315. \tag{2.125}$$

By dividing equations (2.122) and (2.123) by equation (2.124) (or equivalently (2.125)) we obtain

$$u^{-2} + v^{-2} = 21/4, \tag{2.126}$$
$$v^{-1} = 3 - u^{-1}, \tag{2.127}$$

then

$$u^2 - \frac{8}{5}u + \frac{8}{15} = 0. \tag{2.128}$$

Note that u and v are solution of the same equation. We choose arbitrarily $v > u$:

$$u = \frac{4}{5} - \frac{2}{5}\sqrt{\frac{2}{3}} \simeq 0.473401367629, \tag{2.129}$$
$$v = \frac{4}{5} + \frac{2}{5}\sqrt{\frac{2}{3}} \simeq 1.12659863237, \tag{2.130}$$

and finally, using equations (2.124) and (2.125), we have

$$\kappa_{s_1} = \frac{18 + 19\sqrt{2/3}}{280} \simeq 0.11969083942, \tag{2.131}$$

$$\kappa_{s_2} = \frac{18 - 19\sqrt{2/3}}{280} \simeq 0.00888058915143 \tag{2.132}$$

and

$$\kappa_0 = 1 - 12\frac{18}{280} = \frac{8}{35} \simeq 0.228571428571 \tag{2.133}$$

Chapter 3

The equations of motion

3.1 The basic evolution equations

The shape of any ellipsoid can be represented by a 3×3 symmetric matrix $\mathcal{A}$ which may be defined as follows

$$\mathcal{A} = \mathcal{M}\mathcal{D}\mathcal{M}^T \tag{3.1}$$

where $\mathcal{M}$ is an orthonormal rotation matrix whose columns are defined by the three unit vectors ($\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$) lying along the three orthogonal axes of the ellipsoid,

$$\mathcal{M} = \left(\hat{\mathbf{a}} \;\; \hat{\mathbf{b}} \;\; \hat{\mathbf{c}} \right) \tag{3.2}$$

and where $\mathcal{D}$ is a diagonal matrix whose diagonal terms are the inverse squared axis half-lengths:

$$\mathcal{D} = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix} \tag{3.3}$$

The equation for the surface of the ellipsoid reads:

$$\mathbf{x}^T \mathcal{A} \mathbf{x} = 1. \tag{3.4}$$

supposing here that the ellipsoid is centered at the origin.

Taking the time derivative of this shape equation and assuming that the velocity field is linear on the surface of the ellipsoid we have

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u} = \mathcal{S}\mathbf{x}, \tag{3.5}$$

$$\frac{\mathrm{d}\mathbf{x}^T}{\mathrm{d}t}\mathcal{A}\mathbf{x} + \mathbf{x}^T\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\mathbf{x} + \mathbf{x}^T\mathcal{A}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = 0, \tag{3.6}$$

$$\mathbf{x}^T\left(\mathcal{S}^T\mathcal{A} + \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} + \mathcal{A}\mathcal{S}\right)\mathbf{x} = 0. \tag{3.7}$$

As it must hold at any point on the surface of the ellipsoid, we deduce that

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t} = -(\mathcal{S}^T\mathcal{A} + \mathcal{A}\mathcal{S}). \tag{3.8}$$

Note that only a linear velocity field induces only ellipsoidal deformations.

Alternatively, we can rewrite the evolution equation in terms of $\mathcal{B} = \mathcal{A}^{-1}$ by noting that:

$$\mathcal{A}\mathcal{B} = \mathcal{I}, \tag{3.9}$$

$$\mathcal{A}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t} = -\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\mathcal{B}, \tag{3.10}$$

$$\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t} = -\mathcal{B}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}\mathcal{B}, \tag{3.11}$$

$$\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t} = \mathcal{B}(\mathcal{S}^T\mathcal{A} + \mathcal{A}\mathcal{S})\mathcal{B}, \tag{3.12}$$

$$\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t} = \mathcal{B}\mathcal{S}^T + \mathcal{S}\mathcal{B}. \tag{3.13}$$

This is the formulation used in ELM. Note that since $\mathcal{M}$ is orthogonal

$$\mathcal{B} = \mathcal{A}^{-1} = (\mathcal{M}\mathcal{D}\mathcal{M}^T)^{-1} = \mathcal{M}\mathcal{E}\mathcal{M}^T \tag{3.14}$$

where

$$\mathcal{E} = \mathcal{D}^{-1} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \tag{3.15}$$

Note also that the elements of the $\mathcal{B}$ matrix can be deduced from ellipsoid second-order moments:

$$\mathcal{B}_{1,1} = \frac{5}{V}\iiint x^2\mathrm{d}V, \quad \mathcal{B}_{1,2} = \frac{5}{V}\iiint xy\mathrm{d}V, \tag{3.16}$$

$$\mathcal{B}_{1,3} = \frac{5}{V}\iiint xz\mathrm{d}V, \quad \mathcal{B}_{2,2} = \frac{5}{V}\iiint y^2\mathrm{d}V, \tag{3.17}$$

$$\mathcal{B}_{2,3} = \frac{5}{V}\iiint yz\mathrm{d}V, \quad \mathcal{B}_{3,3} = \frac{5}{V}\iiint z^2\mathrm{d}V. \tag{3.18}$$

The other terms are obtained by symmetry. This symmetry implies that the $\mathcal{B}$-matrix can be expressed as

$$\mathcal{B} = \begin{pmatrix} B_1 & B_2 & B_3 \\ B_2 & B_4 & B_5 \\ B_3 & B_5 & B_6 \end{pmatrix} \tag{3.19}$$

and below it proves useful to work with the vector

$$\mathbf{B} = (B_1, B_2, B_3, B_4, B_5, B_6) \tag{3.20}$$

Note that the vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ and the half-lengths a, b, c of the ellipsoid derive from the eigenproblem:

$$\mathcal{B}\hat{\mathbf{a}} = a^2\hat{\mathbf{a}} \tag{3.21}$$

$$\mathcal{B}\hat{\mathbf{b}} = b^2\hat{\mathbf{b}} \tag{3.22}$$

$$\mathcal{B}\hat{\mathbf{c}} = c^2\hat{\mathbf{c}} \tag{3.23}$$

indeed we have

$$\mathcal{B}\hat{\mathbf{a}} = \mathcal{M}\mathcal{E}\mathcal{M}^T\hat{\mathbf{a}} = \mathcal{M}\mathcal{E}\hat{\mathbf{e}}_1 = a^2\mathcal{M}\hat{\mathbf{e}}_1 = a^2\hat{\mathbf{a}} \tag{3.24}$$

$$\mathcal{B}\hat{\mathbf{b}} = \mathcal{M}\mathcal{E}\mathcal{M}^T\hat{\mathbf{b}} = \mathcal{M}\mathcal{E}\hat{\mathbf{e}}_2 = b^2\mathcal{M}\hat{\mathbf{e}}_2 = b^2\hat{\mathbf{b}} \tag{3.25}$$

$$\mathcal{B}\hat{\mathbf{c}} = \mathcal{M}\mathcal{E}\mathcal{M}^T\hat{\mathbf{c}} = \mathcal{M}\mathcal{E}\hat{\mathbf{e}}_3 = c^2\mathcal{M}\hat{\mathbf{e}}_3 = c^2\hat{\mathbf{c}} \tag{3.26}$$

where $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ are the unit vectors along the original x, y, z axes.

We may also note that from equation (3.14)

$$\mathcal{B} = a^2\hat{\mathbf{a}}\hat{\mathbf{a}}^T + b^2\hat{\mathbf{b}}\hat{\mathbf{b}}^T + c^2\hat{\mathbf{c}}\hat{\mathbf{c}}^T \tag{3.27}$$

3.2 Hamiltonian formulation

The Hamiltonian of the system is the scaled total energy (kinetic plus potential) which is given by:

$$\mathcal{H} = \frac{1}{8\pi}\iiint |\boldsymbol{\nabla}\psi|^2 dV = -\frac{1}{8\pi}\iiint q\psi \mathrm{d}V, \tag{3.28}$$

since $q = \Delta\psi$. Inverting the last expression, we have

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\iiint \frac{q(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}V' \tag{3.29}$$

The evolution of the centroid $(X, Y)$and the shape $(\mathcal{B})$ of a given ellipsoid is given by:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = -\frac{1}{\kappa}\mathcal{L}\frac{\partial\mathcal{H}}{\partial\mathbf{X}} \tag{3.30}$$

$$\mathcal{S} = -\frac{10}{\kappa}\mathcal{L}\frac{\partial\mathcal{H}}{\partial\mathcal{B}} \tag{3.31}$$

with

$$\mathcal{L} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.32}$$

arises from $\mathbf{u} = \mathcal{L}\boldsymbol{\nabla}\psi$ and $\kappa = qV/(4\pi)$ is the 'strength' of the vortex.

Note that the structure of $\mathcal{L}$ implies that the third line of the $\mathcal{S}$ matrix is zero – this is a consequence of the lack of vertical advection. Consequently, we can easily show that $\mathrm{d}B_6/\mathrm{d}t = 0$. In fact, $B_6 = \mathcal{B}_{3,3}$ is the squared half-height of the vortex.

proof - shamefully stolen from Dritschel, Reinaud & McKiver (2003) :

Let suppose first that the flow is linear at the boundary of the vortex (Note that, strictly speaking, only such a flow exactly preserve the ellipsoidal shape for the vortex):

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(t) + \mathcal{S}(t)(\mathbf{x} - \mathbf{X}(t)) \tag{3.33}$$

Therefore, the most general linear QG velocity field derives from a streamfunction of the form

$$\psi(\mathbf{x}, t) = \mathbf{F}(t)\cdot\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathcal{P}(t)\mathbf{x} \tag{3.34}$$

Let us now split ψ into a self-induced part ψ_v and a background part ψ_b with $\boldsymbol{F} = \boldsymbol{F}_b$ and $\mathcal{S} = \mathcal{S}_b$. Ignoring the infinite energy of the background flow, we may write

$$\mathcal{H} = \mathcal{H}_v + \mathcal{H}_i \tag{3.35}$$

where $\mathcal{H}_v$ is the self-induced part of the Hamiltonian,

$$\mathcal{H}_v = \frac{1}{8\pi}\iiint|\boldsymbol{\nabla}\psi_v|^2 dV = -\frac{1}{8\pi}\iiint q\psi_v dV \tag{3.36}$$

(using $q = \nabla^2\psi_v$) and $\mathcal{H}_i$ is the part due to the interaction between the background flow and the vortex,

$$\mathcal{H}_i = \frac{1}{4\pi}\iiint\boldsymbol{\nabla}\psi_v\cdot\boldsymbol{\nabla}\psi_b dV = -\frac{1}{4\pi}\iiint q\psi_b dV\,. \tag{3.37}$$

Let us first consider the interaction term $\mathcal{H}_i$. Substituting ψ_b from (with subscripts b on $\boldsymbol{F}$ and $\mathcal{S}$), we may readily evaluate $\mathcal{H}_i$ as

$$\mathcal{H}_i = -\kappa(\boldsymbol{F}_b\cdot\mathbf{X} + \tfrac{1}{2}\mathbf{X}^T\mathcal{P}_b\mathbf{X}) - \frac{1}{10}\kappa\mathcal{P}_b\otimes\mathcal{B} \tag{3.38}$$

where $\mathcal{P}_b \otimes \mathcal{B} \equiv \sum_j \sum_k \mathcal{P}_{bjk}\mathcal{B}_{jk}$ denotes the scalar product of the two matrices.

It follows that

$$\frac{\partial \mathcal{H}_i}{\partial \mathbf{X}} = -\kappa(\boldsymbol{F}_b + \mathcal{P}_b \mathbf{X}) \tag{3.39}$$

$$\frac{\partial \mathcal{H}_i}{\partial \mathcal{B}} = -\frac{1}{10}\kappa \mathcal{P}_b \tag{3.40}$$

and, we see that

$$\boldsymbol{U} = \boldsymbol{U}_b = -\frac{1}{\kappa}\mathcal{L}\frac{\partial \mathcal{H}_i}{\partial \mathbf{X}} \tag{3.41}$$

$$\mathcal{S} = \mathcal{S}_b = -\frac{10}{\kappa}\mathcal{L}\frac{\partial \mathcal{H}_i}{\partial \mathcal{B}}\,, \tag{3.42}$$

Note in particular that the background streamfunction matrix is given by

$$\mathcal{P}_b = -\frac{10}{\kappa}\frac{\partial \mathcal{H}_i}{\partial \mathcal{B}}\,. \tag{3.43}$$

Next consider the self-induced part of the Hamiltonian, $\mathcal{H}_v$. The streamfunction inside the vortex takes the form

$$\psi_v = C + \tfrac{1}{2}(\mathbf{x} - \mathbf{X})^T \mathcal{P}_v (\mathbf{x} - \mathbf{X}) \tag{3.44}$$

where $\mathcal{P}_v = \mathcal{M}\mathcal{D}\mathcal{M}^T$ and $\mathcal{D}$ is a diagonal matrix with

$$\mathcal{D}_{11} = \kappa R_D(b^2, c^2, a^2) \tag{3.45}$$

$$\mathcal{D}_{22} = \kappa R_D(c^2, a^2, b^2) \tag{3.46}$$

$$\mathcal{D}_{33} = \kappa R_D(a^2, b^2, c^2) \tag{3.47}$$

— R_D being the elliptic integral of the second kind — and where C is given by

$$C = -\frac{3}{2}\kappa R_F(a^2, b^2, c^2) \tag{3.48}$$

— R_F being the elliptic integral of the first kind. This constant is required to match ψ_v at the boundary of the ellipsoid with the decaying outer solution, which tends to $\psi_v = -\kappa/r$ as $r = |\mathbf{x} - \mathbf{X}| \to \infty$. Note that the inner solution is quadratic, so the Hamiltonian $\mathcal{H}_v$ (3.36) can also be readily evaluated:

$$\mathcal{H}_v = \frac{3}{4}\kappa^2 R_F(a^2, b^2, c^2) - \frac{1}{20}\kappa \mathcal{P}_v \otimes \mathcal{B} \tag{3.49}$$

$$= \frac{3}{4}\kappa^2 R_F(a^2, b^2, c^2) - \frac{1}{20}\kappa \mathcal{D} \otimes \mathcal{E} \tag{3.50}$$

$$= \frac{3}{5}\kappa^2 R_F(a^2, b^2, c^2) \tag{3.51}$$

using $a^2 R_D(b^2, c^2, a^2) + b^2 R_D(c^2, a^2, b^2) + c^2 R_D(a^2, b^2, c^2) = 3R_F(a^2, b^2, c^2)$ in the last line. Now, by direct calculation, one may show that

$$\mathcal{P}_v = -\frac{10}{\kappa}\frac{\partial \mathcal{H}_v}{\partial \mathcal{B}}\,. \tag{3.52}$$

This gives us the Hamiltonian structure consistent with the problem.

We can show that the system conserves energy:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}\mathcal{B}_i}{\mathrm{d}t} \otimes \frac{\partial \mathcal{H}}{\partial \mathcal{B}_i} + \frac{\mathrm{d}\mathbf{X}_i}{\mathrm{d}t} \cdot \frac{\partial \mathcal{H}}{\partial \mathbf{X}_i} \tag{3.53}$$

We have

$$\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t} \otimes \frac{\partial \mathcal{H}}{\partial \mathcal{B}} \propto (\mathcal{L}\mathcal{P}\mathcal{B} - \mathcal{B}\mathcal{P}^T\mathcal{L}) \otimes \mathcal{P} \tag{3.54}$$

and both $\mathcal{L}\mathcal{P}\mathcal{B} \otimes \mathcal{P}$ and $\mathcal{B}\mathcal{P}^T\mathcal{L} \otimes \mathcal{P} = 0$ for any symmetric mnatrix $\mathcal{B}$ any matrix $\mathcal{P}$ with $\mathcal{P}_{1,2} = \mathcal{P}_{2,1}$ which is incompressibility.

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \cdot \frac{\partial \mathcal{H}}{\partial \mathbf{X}} \propto (U, V, 0) \cdot (-V, U, 0) = 0 \tag{3.55}$$

In terms of the vector $\mathbf{B}$, $\mathcal{S}$ may be written

$$\mathcal{S}_{1,1} = \frac{5}{\kappa}\frac{\partial \mathcal{H}}{\partial B_2}, \quad \mathcal{S}_{1,2} = \frac{10}{\kappa}\frac{\partial \mathcal{H}}{\partial B_4}, \quad \mathcal{S}_{1,3} = \frac{5}{\kappa}\frac{\partial \mathcal{H}}{\partial B_5}, \tag{3.56}$$

$$\mathcal{S}_{2,1} = -\frac{10}{\kappa}\frac{\partial \mathcal{H}}{\partial B_1}, \quad \mathcal{S}_{2,2} = -\mathcal{S}_{1,1} = -\frac{5}{\kappa}\frac{\partial \mathcal{H}}{\partial B_2}, \quad \mathcal{S}_{2,3} = -\frac{5}{\kappa}\frac{\partial \mathcal{H}}{\partial B_3}, \tag{3.57}$$

$$\mathcal{S}_{3,1} = 0, \quad \mathcal{S}_{3,2} = 0, \quad \mathcal{S}_{3,3} = 0 \tag{3.58}$$

We now focus on the part induced by the other ellipsoids. We restrict here the mathematical formulation to two ellipsoids. For the general case with n_v ellipsoids, we only need to repeat the summation over all interacting vortices. Quantities associated with the first vortex are unprimed while those associated with the second are primed. The point vortices (or singularities) introduced in chapter 1 provide an approximation for the streamfunction and suggest a natural discrete form for the Hamiltonian :

$$\mathcal{H}_b = \sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \frac{\kappa_i \kappa'_j}{|\mathbf{x}_i - \mathbf{x}'_j|} \tag{3.59}$$

where n_v is the number of vortices and n_p is the number of singularities representing each vortex . The formula can be viewed as a quadrature formula to calculate the Hamiltonian.

Note that, according to the chapter 1, the position of the singularities may be written in the generic form

$$\mathbf{x}_i = \mathbf{X}_l + \tilde{y}_i \eta \hat{\mathbf{b}} + \tilde{z}_i \tau \hat{\mathbf{c}} \tag{3.60}$$

where $\tilde{y}_i$ and $\tilde{z}_i$ are sets of constant (i.e. they are independant of l). Indeed, the singularities lies around elliptic rings in the plane $(\hat{\mathbf{b}}, \hat{\mathbf{c}})$ whose half-lengths are respectively $\rho\eta$ and $\rho\tau$ with $\rho = \sqrt{\tilde{y}_i^2 + \tilde{z}_i^2}$. However, η, $\hat{\mathbf{b}}$, τ, and $\hat{\mathbf{c}}$ depend on l. The index l is not mentionned to simplify the writing.

We denote

$$r_{ij}^2 = (x_i - x_j')^2 + (y_i - y_j')^2 + (z_i - z_j')^2 \tag{3.61}$$

We have:

$$\frac{\partial \mathcal{H}}{\partial Y} = -\sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \frac{\kappa_i \kappa_j'}{r_{ij}^3}(y_i - y_j') \tag{3.62}$$

$$\frac{\partial \mathcal{H}}{\partial X} = -\sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \frac{\kappa_i \kappa_j'}{r_{ij}^3}(x_i - x_j') \tag{3.63}$$

Similarly, we have

$$\frac{\partial \mathcal{H}}{\partial B_k} = -\sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \frac{\kappa_i \kappa_j'}{r_{ij}^3}(\mathbf{x}_i - \mathbf{x}_j') \cdot \frac{\partial \mathbf{x}_i}{\partial B_k} \tag{3.64}$$

$$\frac{\partial \mathbf{x}_i}{\partial B_k} = \frac{\partial \mathbf{x}_i - \mathbf{X}_l}{\partial B_k} = \tilde{y}_i \frac{\partial \eta \hat{\mathbf{b}}}{\partial B_k} + \tilde{z}_i \frac{\partial \tau \hat{\mathbf{c}}}{\partial B_k} \tag{3.65}$$

$$\frac{\partial \eta \hat{\mathbf{b}}}{\partial B_k} = \frac{\partial \sqrt{\eta^2}\hat{\mathbf{b}}}{\partial B_k}, \quad \eta \geq 0 \tag{3.66}$$

$$\frac{\partial \eta \hat{\mathbf{b}}}{\partial B_k} = \frac{1}{2\eta}\frac{\partial \eta^2}{\partial B_k}\hat{\mathbf{b}} + \eta \frac{\partial \hat{\mathbf{b}}}{\partial B_k} \tag{3.67}$$

$$\frac{\partial \tau \hat{\mathbf{c}}}{\partial B_k} = \frac{\partial \sqrt{\tau^2}\hat{\mathbf{c}}}{\partial B_k}, \quad \tau \geq 0 \tag{3.68}$$

$$\frac{\partial \tau \hat{\mathbf{c}}}{\partial B_k} = \frac{1}{2\tau}\frac{\partial \tau^2}{\partial B_k}\hat{\mathbf{c}} + \tau \frac{\partial \hat{\mathbf{c}}}{\partial B_k} \tag{3.69}$$

We now calculate these partial derivatives. Starting from the eigenproblem, we have:

$$\frac{\partial \mathcal{B}\hat{\mathbf{a}}}{\partial B_k} = \frac{\partial a^2 \hat{\mathbf{a}}}{\partial B_k} \tag{3.70}$$

$$\frac{\partial \mathcal{B}\hat{\mathbf{b}}}{\partial B_k} = \frac{\partial b^2 \hat{\mathbf{b}}}{\partial B_k} \tag{3.71}$$

$$\frac{\partial \mathcal{B}\hat{\mathbf{c}}}{\partial B_k} = \frac{\partial c^2 \hat{\mathbf{c}}}{\partial B_k} \tag{3.72}$$

$$\mathcal{J}_k \hat{\mathbf{a}} + \mathcal{B}\frac{\partial \hat{\mathbf{a}}}{\partial B_k} = \frac{\partial a^2}{\partial B_k}\hat{\mathbf{a}} + a^2 \frac{\partial \hat{\mathbf{a}}}{\partial B_k} \tag{3.73}$$

$$\mathcal{J}_k \hat{\mathbf{b}} + \mathcal{B}\frac{\partial \hat{\mathbf{b}}}{\partial B_k} = \frac{\partial b^2}{\partial B_k}\hat{\mathbf{b}} + b^2 \frac{\partial \hat{\mathbf{b}}}{\partial B_k} \tag{3.74}$$

$$\mathcal{J}_k \hat{\mathbf{c}} + \mathcal{B}\frac{\partial \hat{\mathbf{c}}}{\partial B_k} = \frac{\partial c^2}{\partial B_k}\hat{\mathbf{c}} + c^2 \frac{\partial \hat{\mathbf{c}}}{\partial B_k} \tag{3.75}$$

where

$$\mathcal{J}_k = \frac{\partial \mathcal{B}}{\partial B_k} \tag{3.76}$$

i.e.

$$\mathcal{J}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.77}$$

$$\mathcal{J}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{J}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.78}$$

Left-multiplying equations (3.73) by $\hat{\mathbf{a}}^T$, (3.74) by $\hat{\mathbf{b}}^T$, and (3.75) by $\hat{\mathbf{c}}^T$ we obtain:

$$\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{a}} = \hat{\mathbf{a}}^T (a^2 \mathcal{I} - \mathcal{B})\frac{\partial \hat{\mathbf{a}}}{\partial B_k} + \frac{\partial a^2}{\partial B_k} \tag{3.79}$$

$$\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{b}} = \hat{\mathbf{b}}^T (b^2 \mathcal{I} - \mathcal{B})\frac{\partial \hat{\mathbf{b}}}{\partial B_k} + \frac{\partial b^2}{\partial B_k} \tag{3.80}$$

$$\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{c}} = \hat{\mathbf{c}}^T (c^2 \mathcal{I} - \mathcal{B})\frac{\partial \hat{\mathbf{c}}}{\partial B_k} + \frac{\partial c^2}{\partial B_k} \tag{3.81}$$

with

$$\hat{\mathbf{a}}^T (a^2 \mathcal{I} - \mathcal{B}) = 0 \tag{3.82}$$

$$\hat{\mathbf{b}}^T (b^2 \mathcal{I} - \mathcal{B}) = 0 \tag{3.83}$$

$$\hat{\mathbf{c}}^T (c^2 \mathcal{I} - \mathcal{B}) = 0 \tag{3.84}$$

since this is exactly the eigenproblem (here transposed).

Finally:

$$\frac{\partial a^2}{\partial B_k} = \hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{a}} \tag{3.85}$$
$$\frac{\partial b^2}{\partial B_k} = \hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{b}} \tag{3.86}$$
$$\frac{\partial c^2}{\partial B_k} = \hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{c}} \tag{3.87}$$

We also multiply equation (3.73) by $\hat{\mathbf{b}}^T$ and $\hat{\mathbf{c}}^T$, equation (3.74) by $\hat{\mathbf{a}}^T$ and $\hat{\mathbf{c}}^T$ and equation (3.75) by $\hat{\mathbf{a}}^T$ and $\hat{\mathbf{b}}^T$ to obtain (using again the eigenproblem):

$$\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{a}} + \eta^2 \hat{\mathbf{b}}^T \frac{\partial \hat{\mathbf{a}}}{\partial B_k} = 0 \tag{3.88}$$
$$\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{a}} + \tau^2 \hat{\mathbf{c}}^T \frac{\partial \hat{\mathbf{a}}}{\partial B_k} = 0 \tag{3.89}$$
$$\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{b}} - \eta^2 \hat{\mathbf{a}}^T \frac{\partial \hat{\mathbf{b}}}{\partial B_k} = 0 \tag{3.90}$$
$$\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{b}} + \xi^2 \hat{\mathbf{c}}^T \frac{\partial \hat{\mathbf{b}}}{\partial B_k} = 0 \tag{3.91}$$
$$\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{c}} - \tau^2 \hat{\mathbf{a}}^T \frac{\partial \hat{\mathbf{c}}}{\partial B_k} = 0 \tag{3.92}$$
$$\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{c}} - \xi^2 \hat{\mathbf{b}}^T \frac{\partial \hat{\mathbf{c}}}{\partial B_k} = 0 \tag{3.93}$$
$$\tag{3.94}$$

where

$$\xi \equiv \sqrt{c^2 - b^2} \tag{3.95}$$

Since the vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ are unit vectors, their derivative are perpendicular to them. We pose

$$\frac{\partial \hat{\mathbf{a}}}{\partial B_k} = \lambda_{a2} \hat{\mathbf{b}} + \lambda_{a3} \hat{\mathbf{c}} \tag{3.96}$$
$$\frac{\partial \hat{\mathbf{b}}}{\partial B_k} = \lambda_{b1} \hat{\mathbf{a}} + \lambda_{b3} \hat{\mathbf{c}} \tag{3.97}$$
$$\frac{\partial \hat{\mathbf{c}}}{\partial B_k} = \lambda_{c1} \hat{\mathbf{a}} + \lambda_{c2} \hat{\mathbf{c}} \tag{3.98}$$

which leads to

$$\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{a}} + \eta^2 \lambda_{a2} = 0 \tag{3.99}$$
$$\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{a}} + \tau^2 \lambda_{a3} = 0 \tag{3.100}$$
$$\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{b}} - \eta^2 \lambda_{b1} = 0 \tag{3.101}$$
$$\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{b}} + \xi^2 \lambda_{b3} = 0 \tag{3.102}$$
$$\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{c}} - \tau^2 \lambda_{c1} = 0 \tag{3.103}$$
$$\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{c}} - \xi^2 \lambda_{c2} = 0 \tag{3.104}$$

so finally

$$\frac{\partial \hat{\mathbf{a}}}{\partial B_k} = -\frac{1}{\eta^2}(\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{a}})\hat{\mathbf{b}} - \frac{1}{\tau^2}(\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{a}})\hat{\mathbf{c}} \tag{3.105}$$
$$\frac{\partial \hat{\mathbf{b}}}{\partial B_k} = \frac{1}{\eta^2}(\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{b}})\hat{\mathbf{a}} - \frac{1}{\xi^2}(\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{b}})\hat{\mathbf{c}} \tag{3.106}$$
$$\frac{\partial \hat{\mathbf{c}}}{\partial B_k} = \frac{1}{\tau^2}(\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{c}})\hat{\mathbf{a}} + \frac{1}{\xi^2}(\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{c}})\hat{\mathbf{b}} \tag{3.107}$$

$$\frac{\partial \eta \hat{\mathbf{b}}}{\partial B_k} = \frac{1}{\eta}\left(\frac{1}{2}(\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{b}} - \hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{a}})\hat{\mathbf{b}} + (\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{b}})\hat{\mathbf{a}}\right) - \frac{\eta}{\xi^2}(\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{b}})\hat{\mathbf{c}} \tag{3.108}$$
$$\frac{\partial \tau \hat{\mathbf{c}}}{\partial B_k} = \frac{1}{\tau}\left(\frac{1}{2}(\hat{\mathbf{c}}^T \mathcal{J}_k \hat{\mathbf{c}} - \hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{a}})\hat{\mathbf{c}} + (\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{c}})\hat{\mathbf{a}}\right) + \frac{\eta}{\xi^2}(\hat{\mathbf{b}}^T \mathcal{J}_k \hat{\mathbf{c}})\hat{\mathbf{b}} \tag{3.109}$$

which complete the set of required equations. Note that, since most of the terms of the $\mathcal{J}_k$ matrices are zero, the scalars $\hat{\mathbf{a}}^T \mathcal{J}_k \hat{\mathbf{a}}$, etc... are explicitely developed in the codes, as follows

$$\mathbf{x}^T \mathcal{J}_1 \mathbf{y} = x_1 y_1, \qquad \mathbf{x}^T \mathcal{J}_2 \mathbf{y} = x_1 y_2 + x_2 y_1, \tag{3.110}$$
$$\mathbf{x}^T \mathcal{J}_3 \mathbf{y} = x_1 y_3 + x_3 y_1, \qquad \mathbf{x}^T \mathcal{J}_4 \mathbf{y} = x_2 y_2, \tag{3.111}$$
$$\mathbf{x}^T \mathcal{J}_5 \mathbf{y} = x_2 y_3 + x_2 y_3, \qquad \mathbf{x}^T \mathcal{J}_6 \mathbf{y} = x_3 y_3. \tag{3.112}$$

3.3 Conserved quantities

3.3.1 Angular impulse

The scaled angular impulse is defined by

$$J = \frac{1}{4\pi} \iiint q \boldsymbol{\rho}^2 \mathrm{d}V \tag{3.113}$$

where $\boldsymbol{\rho} = (x, y)$

For a given ellipsoid :

$$J = \kappa \left(X^2 + Y^2 + \iiint \left(\hat{x}^2 + \hat{y}^2 \right) \mathrm{d}V \right) \tag{3.114}$$

where $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{X}$, i.e. the coordinates relative to the centre of the ellipsoid. But this is simply

$$J = \kappa \left(X^2 + Y^2 + \frac{1}{5} \left(B_1 + B_4 \right) \right) \tag{3.115}$$

3.3.2 Energy

The scaled total energy is the Hamiltonian of the system. The interaction energy is then $\mathcal{H}_b$. The self-induced energy is calculated from

$$\mathcal{H}_v = \frac{1}{8\pi} \iiint |\boldsymbol{\nabla} \psi_v|^2 dV = -\frac{1}{2} \iiint q\psi_v \mathrm{d}V = -\frac{2\pi\kappa}{V} \iiint \psi_v \mathrm{d}V \tag{3.116}$$

Note that this is a volume integral of a scalar function. It therefore does not depend on the reference frame in which it is evaluated. For sake of simplicity we adopt a reference frame in which the ellipsoid is centred at the origin and in standard position. Then

$$\psi_v = \psi_0 + \frac{1}{2}\kappa \left(R_D(b^2, c^2, a^2) x^2 + R_D(c^2, a^2, b^2) y^2 + R_D(a^2, b^2, c^2) z^2 \right) \tag{3.117}$$

with

$$\psi_0 = -\frac{3}{2} \kappa R_F(a^2, b^2, c^2) \tag{3.118}$$

being an integration constant. Finally

$$\mathcal{H}_v = \frac{\pi\kappa^2}{V} \iiint \left(3R_F(a^2, b^2, c^2) - \left(R_D(b^2, c^2, a^2) x^2 + R_D(c^2, a^2, b^2) y^2 + R_D(a^2, b^2, c^2) z^2 \right) \right) \tag{3.119}$$

$$\mathcal{H}_v = \pi\kappa^2 \left(3R_F(a^2, b^2, c^2) - \frac{1}{5} \left(R_D(b^2, c^2, a^2) a^2 + R_D(c^2, a^2, b^2) b^2 + R_D(a^2, b^2, c^2) c^2 \right) \right) \tag{3.120}$$

Chapter 4

Steady States

4.1 Full (ill-posed) problem

We first present the approach used to compute steady, or equilibrium states. The approach is in some ways similar to the one used for computing the equilibrium states using the full equations of Contour Dynamics. The first statement to make is that we aim at solving the equation

$$\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t} = \mathcal{B}\mathcal{S}^T + \mathcal{S}\mathcal{B} = 0 \tag{4.1}$$

for each vortex in the proper rotating reference frame (in which $\mathrm{d}\mathbf{X}/\mathrm{d}t = 0$).

The general form of $\mathcal{S}$ is

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_{1,1} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} \\ \mathcal{S}_{2,1} & -\mathcal{S}_{1,1} & \mathcal{S}_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

The simpliest way to compute steady states is to fix the centroid position of the two vortices and find the two $\mathcal{B}$ matrices such that (4.1) is verified. Equation (4.1) is non-linear since both $\mathcal{S}$ matrices depend on BOTH vortices. We solve the non-linear problem using a linear iterative method. Starting for a guess for an equilibrium state, we first calculate the velocities of the centroid (in the reference frame anchored at the global centroid of the two vortices). This provides us an estimation of the rotation rate of the two vortices:

$$\Omega = \frac{1}{2}\sum_{l=1}^{2}\left(\frac{XV - YU}{X^2 + Y^2} + \frac{X'V' - Y'U'}{X'^2 + Y'^2}\right) \tag{4.3}$$

where $U, V = \mathbf{u}(X, Y)$. We also compute the $\mathcal{S}$ matrix for both vortices. The obove rotation must be included in the matrices since we adopt a reference frame rotating with the vortices. This results in

$$\mathcal{S} = \mathcal{S}_0 - \Omega\mathcal{L} \tag{4.4}$$

where $\mathcal{S}_0$ is the flow matrix without rotation.

Then, letting $\mathcal{S}$ be fixed during the iteration, we calculate $\mathcal{B}$. In other words, during one iteration, we ignore the implicit change of $\mathcal{S}$ while correcting the coefficients of $\mathcal{B}$. As $\mathcal{B}$ is symmetric and $\mathcal{B}_{3,3} = B_6$ does not change in quasi-geostrophic flows there are only 5 independent coefficients $(B_1, B_2, B_3, B_4, B_5)$ for each vortex. Moreover, since the two matrices $\mathcal{S}$ are first calculated for given $\mathcal{B}$ matrices, the equation (4.1) can be solved independently for each vortex.

This equation is equivalent to the following 5×5-system:

$$\begin{pmatrix} \mathcal{S}_{1,1} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} & 0 & 0 \\ \mathcal{S}_{2,1} & 0 & \mathcal{S}_{2,3} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} \\ 0 & 0 & \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ 0 & \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} & \mathcal{S}_{2,3} \\ 0 & 0 & \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -B_6\mathcal{S}_{1,3} \\ 0 \\ -B_6\mathcal{S}_{2,3} \end{pmatrix} \tag{4.5}$$

Let $\tilde{\mathcal{S}}$ be the 5×5 matrix above. We next compute the determinant of $\tilde{\mathcal{S}}$ to check if it can be inverted:

$$det(\tilde{\mathcal{S}}) = \mathcal{S}_{1,1} * \begin{vmatrix} 0 & \mathcal{S}_{2,3} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} \\ 0 & \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{11} & \mathcal{S}_{2,3} \\ 0 & \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{vmatrix} - \mathcal{S}_{2,1} * \begin{vmatrix} \mathcal{S}_{1,2} & \mathcal{S}_{1,3} & 0 & 0 \\ 0 & \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{11} & \mathcal{S}_{2,3} \\ 0 & \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{vmatrix} \tag{4.6}$$

$$\begin{aligned} det(\tilde{\mathcal{S}}) = {} & \mathcal{S}_{1,1}.\mathcal{S}_{2,1} * \begin{vmatrix} \mathcal{S}_{2,3} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} \\ \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{vmatrix} \\ & + \mathcal{S}_{2,1}.\mathcal{S}_{1,1} * \begin{vmatrix} \mathcal{S}_{1,2} & \mathcal{S}_{1,3} & 0 \\ 0 & \mathcal{S}_{1,1} & \mathcal{S}_{1,2} \\ 0 & \mathcal{S}_{2,1} & -\mathcal{S}_{1,1} \end{vmatrix} \end{aligned} \tag{4.7}$$

$$det(\tilde{\mathcal{S}}) = \mathcal{S}_{1,1}.\mathcal{S}_{2,1}.(\mathcal{S}_{1,2}^2.\mathcal{S}_{2,1} + \mathcal{S}_{1,1}^2.\mathcal{S}_{1,2}) + \mathcal{S}_{2,1}.\mathcal{S}_{1,1}.(-\mathcal{S}_{1,1}^2.\mathcal{S}_{1,2} - \mathcal{S}_{1,2}^2.\mathcal{S}_{2.1}) = 0 \tag{4.8}$$

The system cannot be inverted. We have to remove one of the equations and add a new constraint. The natural one is volume conservation, introduced below. First however we look, for a 4×4 sub-system that can be inverted by removing one equation and checking the determinant of the sub-system.

4.2 Extracting a subsystem

4.2.1 Removing the first equation

The system becomes

$$\begin{pmatrix} 0 & \mathcal{S}_{2,3} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} \\ 0 & \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} & \mathcal{S}_{2,3} \\ 0 & \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{pmatrix} \begin{pmatrix} B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix} = \begin{pmatrix} -B_1\mathcal{S}_{2,1} \\ -B_6\mathcal{S}_{1,3} \\ 0 \\ -B_6\mathcal{S}_{2,3} \end{pmatrix} \tag{4.9}$$

$$det(\tilde{\mathcal{S}}) = \mathcal{S}_{2,1}\begin{vmatrix} \mathcal{S}_{2,3} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} \\ \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{vmatrix} = \mathcal{S}_{2,1}\mathcal{S}_{1,2}(\mathcal{S}_{1,2}\mathcal{S}_{2,1} + \mathcal{S}_{1,1}^2) \tag{4.10}$$

which is generally non-zero.

4.2.2 Removing the second equation

The system becomes

$$\begin{pmatrix} \mathcal{S}_{1,1} & \mathcal{S}_{1,3} & 0 & 0 \\ 0 & \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ 0 & 0 & -\mathcal{S}_{1,1} & \mathcal{S}_{2,3} \\ 0 & \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix} = \begin{pmatrix} -B_2\mathcal{S}_{1,2} \\ -B_6\mathcal{S}_{1,3} \\ -B_2\mathcal{S}_{2,1} \\ -B_6\mathcal{S}_{2,3} \end{pmatrix} \tag{4.11}$$

$$det(\tilde{\mathcal{S}}) = \mathcal{S}_{1,1}\begin{vmatrix} \mathcal{S}_{1,1} & 0 & \mathcal{S}_{1,2} \\ 0 & -\mathcal{S}_{1,1} & \mathcal{S}_{2,3} \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{vmatrix} = \mathcal{S}_{1,1}^2(\mathcal{S}_{1,1}^2 + \mathcal{S}_{1,2}\mathcal{S}_{2,1}) \tag{4.12}$$

which is generally non-zero.

4.2.3 Removing the third equation

The system becomes

$$\begin{pmatrix} \mathcal{S}_{1,1} & \mathcal{S}_{1,2} & 0 & 0 \\ \mathcal{S}_{2,1} & 0 & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} \\ 0 & \mathcal{S}_{2,1} & -\mathcal{S}_{1,1} & \mathcal{S}_{2,3} \\ 0 & 0 & 0 & -\mathcal{S}_{1,1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_4 \\ B_5 \end{pmatrix} = \begin{pmatrix} -B_3\mathcal{S}_{1,3} \\ -B_3\mathcal{S}_{2,3} \\ 0 \\ -B_3\mathcal{S}_{2,1} - B_6\mathcal{S}_{2,3} \end{pmatrix} \tag{4.13}$$

$$det(\tilde{\mathcal{S}}) = -\mathcal{S}_{1,1}\begin{vmatrix} \mathcal{S}_{1,1} & \mathcal{S}_{1,2} & 0 \\ \mathcal{S}_{2,1} & 0 & \mathcal{S}_{1,2} \\ 0 & \mathcal{S}_{2,1} & -\mathcal{S}_{1,1} \end{vmatrix} = -\mathcal{S}_{1,1}^2(\mathcal{S}_{1,2}\mathcal{S}_{2,1} - \mathcal{S}_{1,2}\mathcal{S}_{2,1}) = 0 \tag{4.14}$$

The third equation must therefore be retained.

4.2.4 Removing the fourth equation

The system becomes

$$\begin{pmatrix} \mathcal{S}_{1,1} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} & 0 \\ \mathcal{S}_{2,1} & 0 & \mathcal{S}_{2,3} & \mathcal{S}_{1,3} \\ 0 & 0 & \mathcal{S}_{1,1} & \mathcal{S}_{1,2} \\ 0 & 0 & \mathcal{S}_{2,1} & -\mathcal{S}_{1,1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -B_4\mathcal{S}_{1,2} \\ -B_6\mathcal{S}_{1,3} \\ -B_6\mathcal{S}_{1,3} \end{pmatrix} \tag{4.15}$$

$$det(\tilde{\mathcal{S}}) = -\mathcal{S}_{1,2}\begin{vmatrix} \mathcal{S}_{2,1} & \mathcal{S}_{2,3} & \mathcal{S}_{1,3} \\ 0 & \mathcal{S}_{1,1} & \mathcal{S}_{1,2} \\ 0 & \mathcal{S}_{2,1} & 1\mathcal{S}_{1,1} \end{vmatrix} = \mathcal{S}_{1,2}.cS_{2,1}(\mathcal{S}_{1,1}^2 + \mathcal{S}_{2,1}\mathcal{S}_{1,2}) \tag{4.16}$$

which is generally non-zero.

4.2.5 Removing the fifth equation

The system becomes

$$\begin{pmatrix} \mathcal{S}_{1,1} & \mathcal{S}_{1,2} & \mathcal{S}_{1,3} & 0 \\ \mathcal{S}_{2,1} & 0 & \mathcal{S}_{2,3} & \mathcal{S}_{1,2} \\ 0 & 0 & \mathcal{S}_{1,1} & 0 \\ 0 & \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -B_5\mathcal{S}_{1,3} \\ -B_5\mathcal{S}_{1,2} - B_6\mathcal{S}_{1,3} \\ -B_5\mathcal{S}_{2,3} \end{pmatrix} \tag{4.17}$$

$$det(\tilde{\mathcal{S}}) = \mathcal{S}_{1,1}\begin{vmatrix} 0 & \mathcal{S}_{2,3} & \mathcal{S}_{1,2} \\ 0 & \mathcal{S}_{1,1} & 0 \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{vmatrix} - \mathcal{S}_{2,1}\begin{vmatrix} \mathcal{S}_{1,2} & \mathcal{S}_{1,3} & 0 \\ 0 & \mathcal{S}_{1,1} & 0 \\ \mathcal{S}_{2,1} & 0 & -\mathcal{S}_{1,1} \end{vmatrix} \tag{4.18}$$

$$= -\mathcal{S}_{1,1}^2.\mathcal{S}_{1,2}.\mathcal{S}_{2,1} + \mathcal{S}_{1,1}^2.\mathcal{S}_{1,2}.\mathcal{S}_{2,1} = 0 \tag{4.19}$$

he third equation must therefore be retained.

In practice, we have chosen to remove the first equation, but other choices are possible.

4.3 Volume conservation

We now need a fifth equation to complete the system. We enforce volume conservation, at least its linearised approximation. Note that the volume of an ellipsoid is

$$V = \frac{4\pi abc}{3} \tag{4.20}$$

while

$$\det(\mathcal{B}) = \det(\mathcal{E}) = (abc)^2 \tag{4.21}$$

with

$$\det(\mathcal{B}) = B_1B_4B_6 + 2B_2B_5B_3 - B_3^2B_4 - B_5^2B_1 - B_2^2B_6 \tag{4.22}$$

Let's call $\bar{\mathbf{B}}$ the estimate for $\mathbf{B}$ in the previous iteration and $\tilde{\mathbf{B}}$ the correction to add:

$$\begin{array}{ll} B_1 = \bar{B}_1 + \tilde{B}_1 & B_2 = \bar{B}_2 + \tilde{B}_2 \\ B_3 = \bar{B}_3 + \tilde{B}_3 & B_4 = \bar{B}_4 + \tilde{B}_4 \\ B_5 = \bar{B}_5 + \tilde{B}_5 & B_6 = \bar{B}_6 \end{array}$$

To the first order in the corrections, we have:

$$\begin{aligned} \det(\mathcal{B}) \simeq \det(\bar{\mathcal{B}}) + (\bar{B}_4\bar{B}_6 - \bar{B}_5^2)\tilde{B}_1 + 2(\bar{B}_5\bar{B}_3 - \bar{B}_6\bar{B}_2)\tilde{B}_2 \\ +2(\bar{B}_5\bar{B}_2 - \bar{B}_4\bar{B}_3)\tilde{B}_3 + (\bar{B}_1\bar{B}_6 - \bar{B}_3^2)\tilde{B}_4 + 2(\bar{B}_2\bar{B}_3 - \bar{B}_1\bar{B}_5)\tilde{B}_5 = (abc)^2 \end{aligned} \tag{4.24}$$

Reintroducing $\mathbf{B} = \bar{\mathbf{B}} + \tilde{\mathbf{B}}$:

$$\begin{aligned} (\bar{B}_4\bar{B}_6 - \bar{B}_5^2)B_1 + 2(\bar{B}_5\bar{B}_3 - \bar{B}_6\bar{B}_2)B_2 + 2(\bar{B}_5\bar{B}_2 - \bar{B}_4\bar{B}_3)B_3 \\ +(\bar{B}_1\bar{B}_6 - \bar{B}_3^2)B_4 + 2(\bar{B}_2\bar{B}_3 - \bar{B}_1\bar{B}_5)B_5 = (abc)^2 - \bar{B}_6\bar{B}_2^2 \\ +2\bar{B}_2\bar{B}_3\bar{B}_5 - \bar{B}_4\bar{B}_3^2 + \bar{B}_1\bar{B}_6\bar{B}_4 + 2\bar{B}_2\bar{B}_3\bar{B}_4 - 2\bar{B}_1\bar{B}_5^2 \end{aligned} \tag{4.25}$$

which provides the fifth equation for the system.

This scheme works for any strength ratio between the two vortices EXCEPT when the ratio is exactly -1. In this situation, the vortices do not rotate but translate (in fact the center of the rotation which the global centroid is at infinity). We have to slightly modify the existing code for this particular configuration.

4.4 Using the gap as a parameter

In the previous version, the centroid positions are fixed and we iterate on the $\mathcal{B}$ matrices. Nevertheless, when dealing with like-signed vortices the centroid separation is not monotonic when looking at a full branch of solutions. The gap however is monotonic in all cases studied. We have developed a new set of equations for which the gap is fixed and both the centroid positions and $\mathcal{B}$ are modified.

In practice, we know that we can neglect the implicit change of $\mathcal{S}$ while correcting $\mathcal{B}$ during an iteration. However, this is not possible for $\mathbf{X}$. Neglecting the variations of $\mathcal{S}$ as $\mathbf{X}$ is corrected causes the procedure to diverge. Besides $\mathrm{d}\mathcal{S}_b(\text{vortex}\,1)/\mathrm{d}\mathbf{X}(\text{vortex}\,2) \neq 0$. The problem is now coupled for the two vortices. We again use the unprimed/primed notations to distinguish vortex 1 form vortex 2.

The vortices are aligned along the x-axis (i.e. $Y = Y' = 0$). By convention, vortex 2 has a positive abscissa, vortex 1 has a negative one (in the cases where this algorithm is used — i.e. co-rotating vortices)

The centroid position correction is linked to the change in the $\mathcal{B}$ matrices since we want to impose the gap. The gap between the two vortices is

$$\Delta = (X' - \sqrt{B_1'}) - (X - \sqrt{B_1}) \tag{4.26}$$

The relationship between the gap and B_1, B_1' is non-linear. We therefore linearize it with respect to the correction $\tilde{B}_1$ and $\tilde{B}_1'$. This is the second main difference with the previous version of the procedure. The matrix problem to solve will now be expressed in terms of the corrections on $\mathbf{B}$ rather than in terms of the $\mathbf{B}$ coefficients themselves.

We write:

$$B_1 = (X - \Delta)^2 \tag{4.27}$$

$$\bar{B}_1 + \tilde{B}_1 = (\bar{X} + \tilde{X} - \Delta)^2 \tag{4.28}$$

$$\bar{B}_1 + \tilde{B}_1 = (\bar{X} - \Delta)^2 + 2(\bar{X} - \Delta)\tilde{X} \tag{4.29}$$

$$\tilde{X} = \frac{\bar{B}_1 - (\bar{X} - \Delta)^2}{2(\bar{X} - \Delta)} + \frac{\tilde{B}_1}{2(\bar{X} - \Delta)} \tag{4.30}$$

respectively,

$$\tilde{X}' = \frac{\bar{B}_1' - (\bar{X}' - \Delta)^2}{2(\bar{X}' - \Delta)} + \frac{\tilde{B}_1'}{2(\bar{X}' - \Delta)} \tag{4.31}$$

We now derive the linearised equations to solve by expanding (4.1) as follows:

$$(\bar{\mathcal{B}} + \tilde{\mathcal{B}}) \left(\mathcal{S}^T + \frac{\partial \mathcal{S}^T}{\partial X}\tilde{X} + \frac{\partial \mathcal{S}^T}{\partial X'}\tilde{X}' \right) + \left(\mathcal{S}^T + \frac{\partial \mathcal{S}}{\partial X}\tilde{X} + \frac{\partial \mathcal{S}_i}{\partial X'}\tilde{X}' \right) (\bar{\mathcal{B}} + \tilde{\mathcal{B}}) = 0 \tag{4.32}$$

$$(\bar{\mathcal{B}}' + \tilde{\mathcal{B}}') \left(\mathcal{S}'^T + \frac{\partial \mathcal{S}'^T}{\partial X'}\tilde{X}' + \frac{\partial \mathcal{S}'^T}{\partial X}\tilde{X} \right) + \left(\mathcal{S}'^T + \frac{\partial \mathcal{S}'}{\partial X'}\tilde{X}' + \frac{\partial \mathcal{S}'}{\partial X}\tilde{X} \right) (\bar{\mathcal{B}}' + \tilde{\mathcal{B}}') = 0 \tag{4.33}$$

where $\mathcal{S}$, $\mathcal{S}'$ and their derivatives are evaluated using $\bar{\mathbf{X}}$, $\bar{\mathbf{X}}'$, $\bar{\mathcal{B}}$ and $\bar{\mathcal{B}}'$.

Recall that $\mathcal{S} = -\dfrac{10}{\kappa}\mathcal{L}\dfrac{\partial \mathcal{H}}{\partial \mathcal{B}}$. Then, since

$$\frac{\partial^2 \mathcal{H}}{\partial X \partial B_k} = -\sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \kappa_i \kappa_j' \frac{\partial \mathbf{x}_i}{\partial B_k} \cdot \left(\frac{\hat{\mathbf{e}}_1}{r_{ij}^3} - 3\frac{(x_i - x_j')(\mathbf{x}_i - \mathbf{x}_j')}{r_{ij}^5} \right) \tag{4.34}$$

and

$$\frac{\partial^2 \mathcal{H}}{\partial X' \partial B_k} = -\frac{\partial^2 \mathcal{H}}{\partial X \partial B_k} \tag{4.35}$$

and similarly for $\dfrac{\partial^2 \mathcal{H}}{\partial X' \partial B_k'}$ by swapping unprimed and primed notations, we have

$$\frac{\partial S}{\partial X} = -\frac{\partial S}{\partial X'}, \quad \frac{\partial S'}{\partial X} = -\frac{\partial S'}{\partial X'} \tag{4.36}$$

We denote

$$\lambda = \frac{1}{2(\bar{X} - \Delta)}, \quad \lambda' = \frac{1}{2(\bar{X}' - \Delta)} \tag{4.37}$$

$$\alpha = \frac{B_1 - (\bar{X} - \Delta)^2}{2(\bar{X} - \Delta)}, \quad \alpha' = \frac{B_1' - (X' - \Delta)^2}{2(\bar{X}' - \Delta)} \tag{4.38}$$

$$\mathcal{T} = \mathcal{S} + \alpha \frac{\partial \mathcal{S}}{\partial X} - \alpha' \frac{\partial \mathcal{S}}{\partial X}, \quad \mathcal{T}' = \mathcal{S}' + \alpha' \frac{\partial \mathcal{S}'}{\partial X'} - \alpha \frac{\partial \mathcal{S}'}{\partial X} \tag{4.39}$$

$$\mathcal{U} = \bar{\mathcal{B}} \frac{\partial \mathcal{S}^T}{\partial X} + \frac{\partial \mathcal{S}}{\partial X} \bar{\mathcal{B}}, \quad \mathcal{U}' = \bar{\mathcal{B}}' \frac{\partial \mathcal{S}'^T}{\partial X'} + \frac{\partial \mathcal{S}'}{\partial X'} \bar{\mathcal{B}}' \tag{4.40}$$

Then, the equations are:

$$\tilde{\mathcal{B}} \mathcal{T}^T + \mathcal{T} \tilde{\mathcal{B}}_1 + \tilde{B}_1 \lambda \mathcal{U} - \tilde{B}_1' \lambda' \mathcal{U} = -\bar{\mathcal{B}} \mathcal{T}^T - \mathcal{T} \bar{\mathcal{B}} \tag{4.41}$$

$$\tilde{\mathcal{B}}' \mathcal{T}'^T + \mathcal{T}' \tilde{\mathcal{B}}' + \tilde{B}_1 \lambda' \mathcal{U}' - \tilde{B}_1' \lambda \mathcal{U}' = -\bar{\mathcal{B}}' \mathcal{T}'^T - \mathcal{T}' \bar{\mathcal{B}}' \tag{4.42}$$

This can be recast in a 10×10-matrix problem.

$$\mathcal{C} \tilde{\mathbf{B}}^* = \mathbf{D}^* \tag{4.43}$$

where

$$\tilde{\mathbf{B}}^* = (\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4, \tilde{B}_5, \tilde{B}_1', \tilde{B}_2', \tilde{B}_3', \tilde{B}_4', \tilde{B}_5') \tag{4.44}$$

and

$$\mathcal{C} = \begin{pmatrix} 2\mathcal{T}_{1,1} + \lambda \mathcal{U}_{1,1} & 2\mathcal{T}_{1,2} & 2\mathcal{T}_{1,3} & 0 & 0 & -\lambda' \mathcal{U}_{1,1} & 0 & 0 & 0 & 0 \\ \mathcal{T}_{2,1} + \lambda \mathcal{U}_{1,2} & 0 & \mathcal{T}_{2,3} & \mathcal{T}_{1,2} & \mathcal{T}_{1,3} & -\lambda' \mathcal{U}_{1,2} & 0 & 0 & 0 & 0 \\ \lambda \mathcal{U}_{1,3} & 0 & \mathcal{T}_{1,1} & 0 & \mathcal{T}_{1,2} & -\lambda' \mathcal{U}_{1,3} & 0 & 0 & 0 & 0 \\ \lambda \mathcal{U}_{2,2} & 2\mathcal{T}_{2,1} & 0 & -2\mathcal{T}_{1,1} & 2\mathcal{T}_{2,3} & -\lambda' \mathcal{U}_{2,2} & 0 & 0 & 0 & 0 \\ \lambda \mathcal{U}_{2,3} & & 0\mathcal{T}_{2,1} & 0 & -\mathcal{T}_{1,1} & -\lambda' \mathcal{U}_{2,3} & 0 & 0 & 0 & 0 \\ -\lambda \mathcal{U}_{1,1}' & 0 & 0 & 0 & 0 & 2\mathcal{T}_{1,1}' + \lambda' \mathcal{U}_{1,1}' & 2\mathcal{T}_{1,2}' & 2\mathcal{T}_{1,3}' & 0 & 0 \\ -\lambda \mathcal{U}_{1,2}' & 0 & 0 & 0 & 0 & \mathcal{T}_{2,1}' + \lambda' \mathcal{U}_{1,2}' & 0 & \mathcal{T}_{2,3}' & \mathcal{T}_{1,2}' & \mathcal{T}_{1,3}' \\ -\lambda \mathcal{U}_{1,3}' & 0 & 0 & 0 & 0 & \lambda' \mathcal{U}_{1,3}' & 0 & \mathcal{T}_{1,1}' & 0 & \mathcal{T}_{1,2}' \\ -\lambda \mathcal{U}_{2,2}' & 0 & 0 & 0 & 0 & \lambda' \mathcal{U}_{2,2}' & 2\mathcal{T}_{2,1}' & 0 & -2\mathcal{T}_{1,1}' & 2\mathcal{T}_{2,3}' \\ -\lambda \mathcal{U}_{2,3}' & 0 & 0 & 0 & 0 & \lambda' \mathcal{U}_{2,3}' & & 0\mathcal{T}_{2,1}' & 0 & -\mathcal{T}_{1,1}' \end{pmatrix} \tag{4.45}$$

and

$$
\mathbf{D}^* = \begin{pmatrix}
2(\mathcal{T}_{1,1}\bar{B}_1 + \mathcal{T}_{1,2}\bar{B}_2 + \mathcal{T}_{1,3}\bar{B}_3) \\
\mathcal{T}_{2,1}\bar{B}_1 + \mathcal{T}_{2,3}\bar{B}_3 + \mathcal{T}_{1,2}\bar{B}_4 + \mathcal{T}_{1,3}\bar{B}_5 \\
\mathcal{T}_{1,1}\bar{B}_3 + \mathcal{T}_{1,2}\bar{B}_5 + \mathcal{T}_{1,3}\bar{B}_6 \\
2(\mathcal{T}_{2,1}\bar{B}_2 - \mathcal{T}_{1,1}\bar{B}_4 + \mathcal{T}_{2,3}\bar{B}_5) \\
\mathcal{T}_{2,1}\bar{B}_3 - \mathcal{T}_{1,1}\bar{B}_5 + \mathcal{T}_{2,3}\bar{B}_6 \\
2(\mathcal{T}'_{1,1}\bar{B}'_1 + \mathcal{T}'_{1,2}\bar{B}'_2 + \mathcal{T}'_{1,3}\bar{B}'_3) \\
\mathcal{T}'_{2,1}\bar{B}'_1 + \mathcal{T}'_{2,3}\bar{B}'_3 + \mathcal{T}'_{1,2}\bar{B}'_4 + \mathcal{T}'_{1,3}\bar{B}'_5 \\
\mathcal{T}'_{1,1}\bar{B}'_3 + \mathcal{T}'_{1,2}\bar{B}'_5 + \mathcal{T}'_{1,3}\bar{B}'_6 \\
2(\mathcal{T}'_{2,1}\bar{B}'_2 - \mathcal{T}'_{1,1}\bar{B}'_4 + \mathcal{T}'_{2,3}\bar{B}'_5) \\
\mathcal{T}'_{2,1}\bar{B}'_3 - \mathcal{T}'_{1,1}\bar{B}'_5 + \mathcal{T}'_{2,3}\bar{B}'_6
\end{pmatrix} \tag{4.46}
$$

Again, two equations are suppressed (the first and sixth ones) and are replaced by volume conservation for both vortices. In terms of corrections, these equations read:

$$
\begin{aligned}
&(\bar{B}_4\bar{B}_6 - \bar{B}_5^2)\tilde{B}_1 + 2(\bar{B}_5\bar{B}_3 - \bar{B}_6\bar{B}_2)\tilde{B}_2 + 2(\bar{B}_5\bar{B}_2 - \bar{B}_4\bar{B}_3)\tilde{B}_3 \\
&+(\bar{B}_1\bar{B}_6 - \bar{B}_3^2)\tilde{B}_4 + 2(\bar{B}_2\bar{B}_3 - \bar{B}_1\bar{B}_5)\tilde{B}_5 = (abc)^2 - \det(\mathcal{B})
\end{aligned} \tag{4.47}
$$

and similarly for $\tilde{\mathcal{B}}'$.