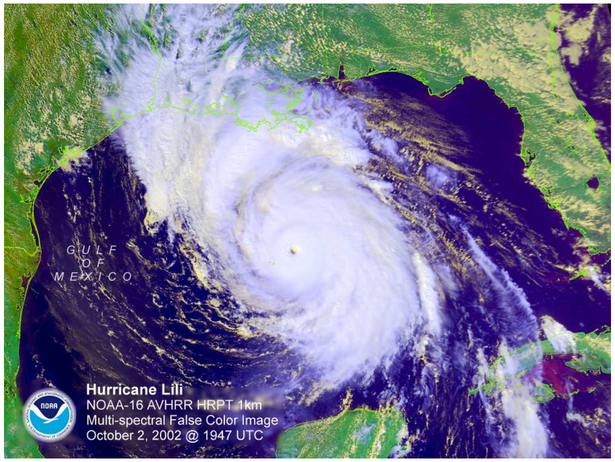
Contour-advective semi-Lagrangian algorithms for the spherical shallow water equations

Ali R Mohebalhojeh
&
David G Dritschel

School of Mathematics
University of St. Andrews

arm@mcs.st-and.ac.uk

- Two distinct types of motion co-exist:
  - ◇ "Slow" balanced vortical motions, and
  - ◇ Relatively "fast" imbalanced wave motions



⇒ which however are deeply intertwined

The shallow-water equations

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\boldsymbol{k} \times \boldsymbol{u} = -c^2 \boldsymbol{\nabla} \tilde{h}$$

$$\frac{\partial \tilde{h}}{\partial t} + \boldsymbol{\nabla} \cdot [(1 + \tilde{h})\boldsymbol{u}] = 0$$

where $\tilde{h} \equiv (h - H)/H$, $c^2 = gH$, $f = 2\Omega_E \sin\phi$, H is the mean depth, g is gravity, and $\boldsymbol{u}$ is tangent to the sphere.

Dissipation and forcing terms are not included.

These equations may be combined to show

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0, \quad \text{where} \quad Q = \frac{\zeta + f}{1 + \tilde{h}}$$

is the potential vorticity (PV).
NB: $\zeta = \boldsymbol{k} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u})$.

Why another shallow-water model?

- Explicit potential-vorticity (PV) conservation has never been implemented in spherical geometry.
- Wave–vortex decomposition not well understood even in this simple context.
- Accurate modelling of *both* the PV-controlled balanced flow and the imbalanced flow is now possible.

New variables

The original equations hide PV conservation
⇒ numerically, PV is poorly conserved.

The distribution of PV largely controls the fluid motion $(\boldsymbol{u}, \tilde{h})$ through hidden balance relations (PV inversion).
— Hoskins, McIntyre & Robertson (1985),
McIntyre & Norton (1999), Ford, McIntyre & Norton (2000), etc.

⇒ numerically, a poor representation of the PV leads to a poor representation of the fluid motion.

⇒ The residual motion, the "imbalance" (gravity waves), may be prone to **large** errors.

A new approach

- ◇ Enforce PV conservation explicitly
  (preserve its advective character)
  ⇒ use *contour advection*;

- ◇ Distinguish the PV-controlled balanced
  motions and the residual imbalanced
  motions, at least to leading order
  ⇒ use *imbalanced prognostic variables*.

— Dritschel & Mohebalhojeh (2000), M & D (2000,2001,2004) for the SW equations, D & Viúdez (2003), V & D (2003,2004) for the nonhydrostatic Boussinesq eqns.

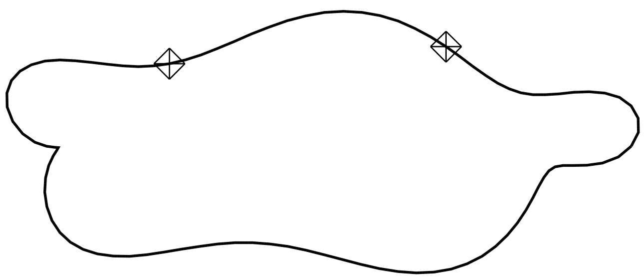
Explicit PV conservation

A particle representation for PV is natural:

each particle $\boldsymbol{x} = \boldsymbol{X}$ conserves its value of Q

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{X}, t)$$

A contour representation is even more natural, since exchanging any pair of particles on a contour Q = constant does not alter the distribution of Q



Numerics: an **ideal** algorithm?

The Contour-Advective Semi-Lagrangian (CASL) algorithm (Dritschel & Ambaum, 1997) makes direct use of this contour representation, *and* deals with the non-locality (inversion) efficiently.

It represents the PV by a finite set of contours

— the Lagrangian aspect —,

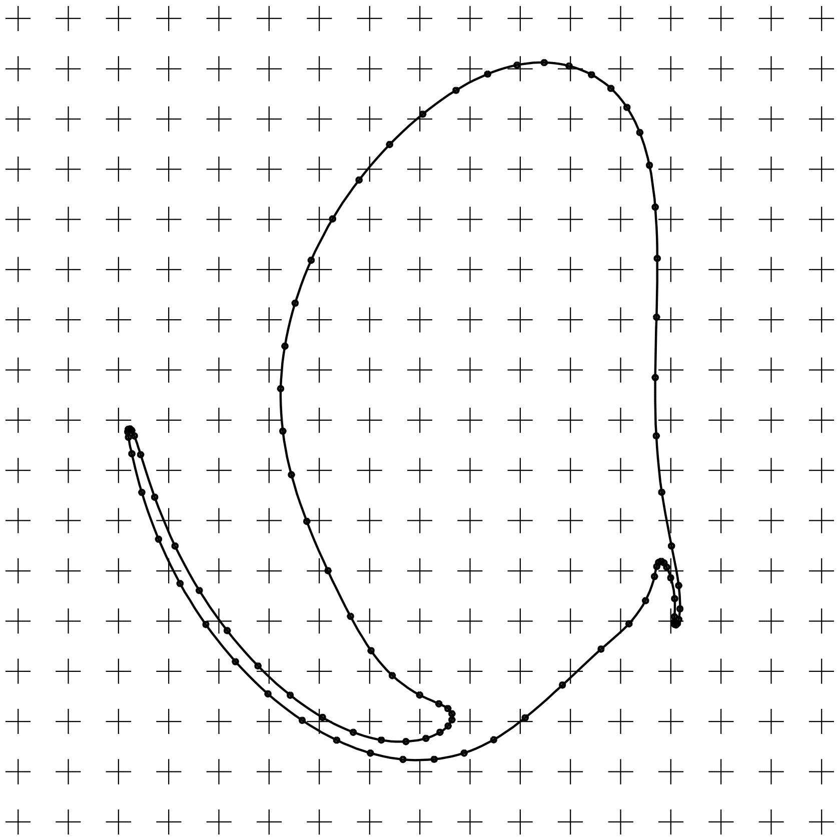
represents the velocity by fixed grid points

— the Eulerian aspect —,

and provides efficient means of communication between the two representations

— interpolation, and its inverse, filling.

Each PV contour is represented by *nodes*, connected together by cubic splines. Shown also is the underlying grid.



Note that Q is permitted to have much finer structure than $\boldsymbol{u}$. This exploits the fact that $\boldsymbol{u}$ is typically a *smoother* field than Q.

Balance relations

Having chosen the PV as one prognostic variable, what is a sensible (accurate *and* convenient) choice for the other two?

In a *balanced model*, the fluid motion $(\boldsymbol{u}_{\mathsf{b}}, \tilde{h}_{\mathsf{b}})$ is fully controlled by the PV.
The balanced motion is recovered by PV inversion, i.e. by solving equations of the form

$$\mathcal{F}(\boldsymbol{u}_{\mathsf{b}}, \tilde{h}_{\mathsf{b}}) = 0, \quad \mathcal{G}(\boldsymbol{u}_{\mathsf{b}}, \tilde{h}_{\mathsf{b}}) = 0, \quad \mathcal{H}(\boldsymbol{u}_{\mathsf{b}}, \tilde{h}_{\mathsf{b}}) = 0$$

for $\boldsymbol{u}_{\mathsf{b}}$ and $\tilde{h}_{\mathsf{b}}$, *given* the PV Q.

One of these equations comes from the definition of PV:

$$\mathcal{F} = \boldsymbol{k} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}_{\mathsf{b}}) + f - Q(1 + \tilde{h}_{\mathsf{b}}) = 0$$

The other two come from imposing particular relations between variables, e.g. as in geostrophic balance.

For example, one may set two successive time derivatives of the divergence $\delta = \boldsymbol{\nabla} \cdot \boldsymbol{u}$ to be zero, i.e.

$$\mathcal{G} = \delta^{(n)} = 0, \quad \mathcal{H} = \delta^{(n+1)} = 0$$

generating the "δ hierarchy". The forms of $\mathcal{G}$ and $\mathcal{H}$ are found by recursively substituting the original equations (M & D 2001).

Another example, used here, sets

$$\mathcal{G} = \delta^{(n)} = 0, \quad \mathcal{H} = \gamma^{(n)} = 0$$

where $\gamma = \boldsymbol{\nabla} \cdot \boldsymbol{a}$ and $\boldsymbol{a} = \mathrm{D}\boldsymbol{u}/\mathrm{D}t$.

NB: $\boldsymbol{a} = -f\boldsymbol{k} \times \boldsymbol{u} - c^2\boldsymbol{\nabla}\tilde{h}$ is the acceleration

It makes sense that the new variables should represent what the PV cannot.

$\Rightarrow$ The new variables could be $\mathcal{G}$ and $\mathcal{H}$ themselves.

Here, we use the simplest member, $n = 0$, of the δ-γ hierarchy. In other words, we take

$$\mathcal{G} = \delta, \qquad \mathcal{H} = \gamma$$

to be the other two prognostic variables.

- On the f-plane, γ is proportional the "ageostrophic vorticity", $\zeta - c^2\nabla^2\tilde{h}/f$.

  Setting $\gamma = \delta = 0$ then leads to geostrophic balance (cf. M & D 2001). The variables γ and δ thus represent the *departure* from geostrophic balance.

- On the sphere,

$$\gamma = f\zeta - \beta u - c^2\nabla^2\tilde{h}$$

  where $\beta = \mathrm{d}f/\mathrm{d}\phi = 2\Omega_E \cos\phi$ and u is the zonal velocity component.

The prognostic equations for δ and γ are

$$\frac{\partial \delta}{\partial t} = \gamma - |\boldsymbol{u}|^2 - 2\left[\frac{\partial u}{\partial \phi}\left(\frac{\partial u}{\partial \phi} + \zeta\right) + \frac{\partial v}{\partial \phi}\left(\frac{\partial v}{\partial \phi} - \delta\right)\right] - \boldsymbol{\nabla}\cdot(\delta \boldsymbol{u})$$

$$\frac{\partial \gamma}{\partial t} = c^2\nabla^2\{\boldsymbol{\nabla}\cdot[(1+\tilde{h})\boldsymbol{u}]\} + 2\Omega_E\frac{\partial B}{\partial \lambda} - \boldsymbol{\nabla}\cdot(Z\boldsymbol{u})$$

where $B \equiv c^2\tilde{h} + \frac{1}{2}|\boldsymbol{u}|^2$ (Bernoulli pressure), $Z = f(\zeta + f)$, and λ is longitude.

However, the tendencies involve the original variables $\boldsymbol{u}$ and $\tilde{h}$. These are recovered by a kind of PV inversion analogous to what is done in a balanced model.

Inversion

Inversion here simply means finding $\boldsymbol{u}$ and $\tilde{h}$ from the prognostic variable set (Q, δ, γ).

This is accomplished as follows. Let

$$\boldsymbol{u} = \boldsymbol{k} \times \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\chi$$

then the potentials satisfy

$$\nabla^2\psi = \zeta \qquad \& \qquad \nabla^2\chi = \delta\,.$$

But ζ depends on $\tilde{h}$ through the definition of PV:

$$\zeta = (1+\tilde{h})Q - f\,.$$

So, we need to find $\tilde{h}$ before we can invert ζ. But the definition of γ implies

$$c^2\nabla^2\tilde{h} - fQ\tilde{h} = f(Q-f) - \beta u - \gamma\,,$$

using ζ above.

While the inversion equations are coupled, they are *linear*, an exceptional property.

Numerically, they are solved iteratively and convergence is exponentially fast.

Numerics

◇ All fields represented on a regular lat-lon grid, with $\Delta\phi = \Delta\lambda/2 \quad (n_\phi = n_\lambda)$

◇ Semi-spectral approach: advantageous for inverting $c^2\nabla^2 - f^2$ (tridiagonal procedure)

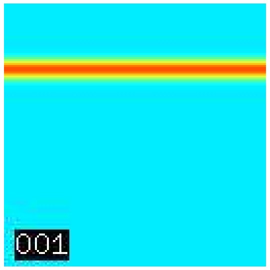
◇ 2nd-order centred, 4th-order compact, and 6th-order super compact (QJRMS 2005, 2109–2129) finite differences in ϕ

◇ Semi-implicit time stepping, but

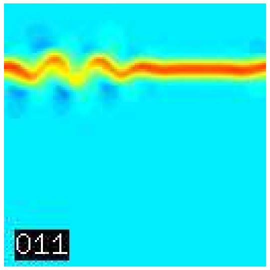
$$\Delta t < \Delta t_{\mathrm{CFL}} = \Delta\phi/c$$

◇ **Numerical experiments:** Galewsky, Scott & Polvani test case: Tellus (2004), 429–440

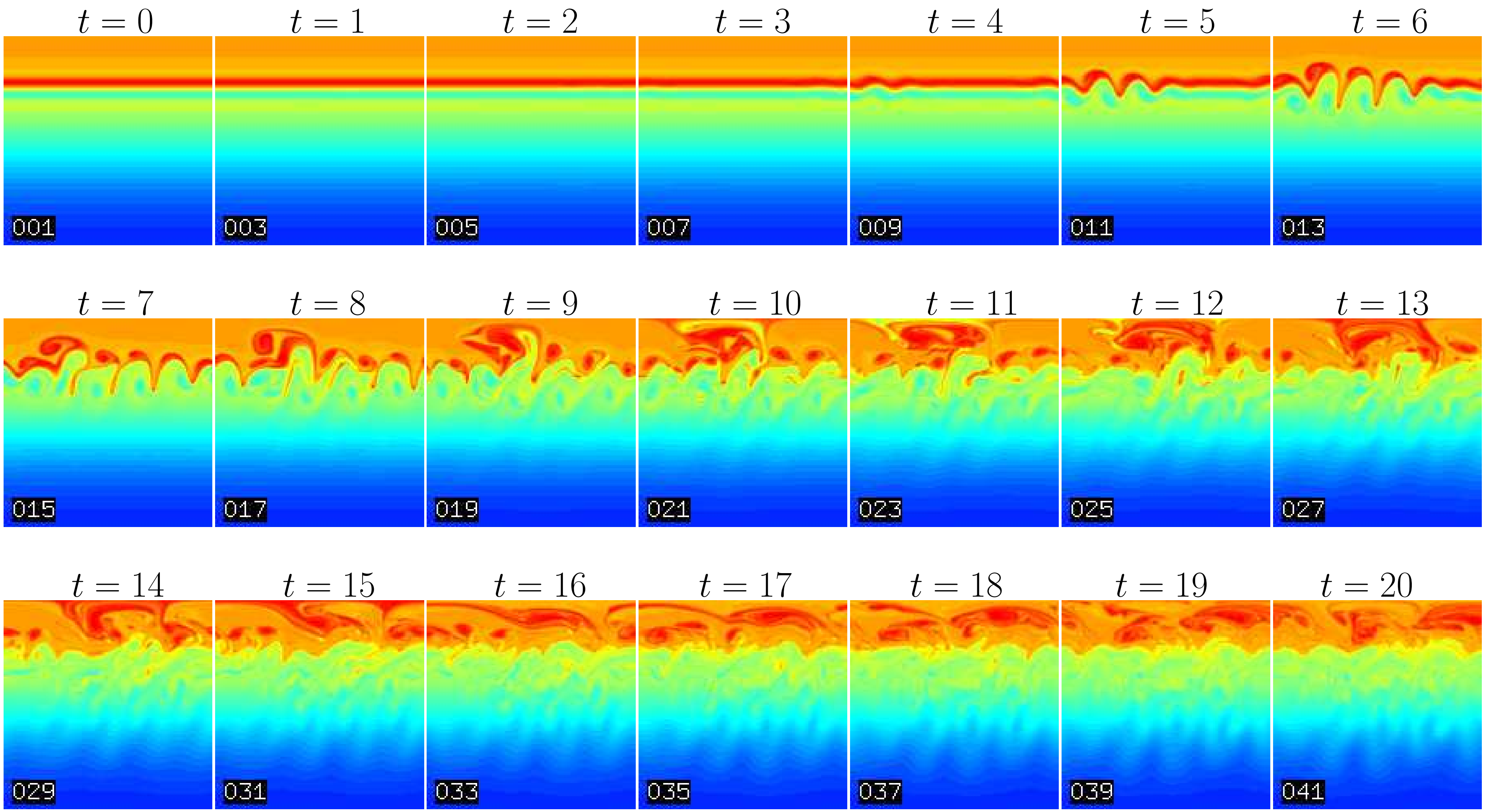
$t = 0.0$

◇ **A mid-latitude zonal jet:** 

$t = 5.0$

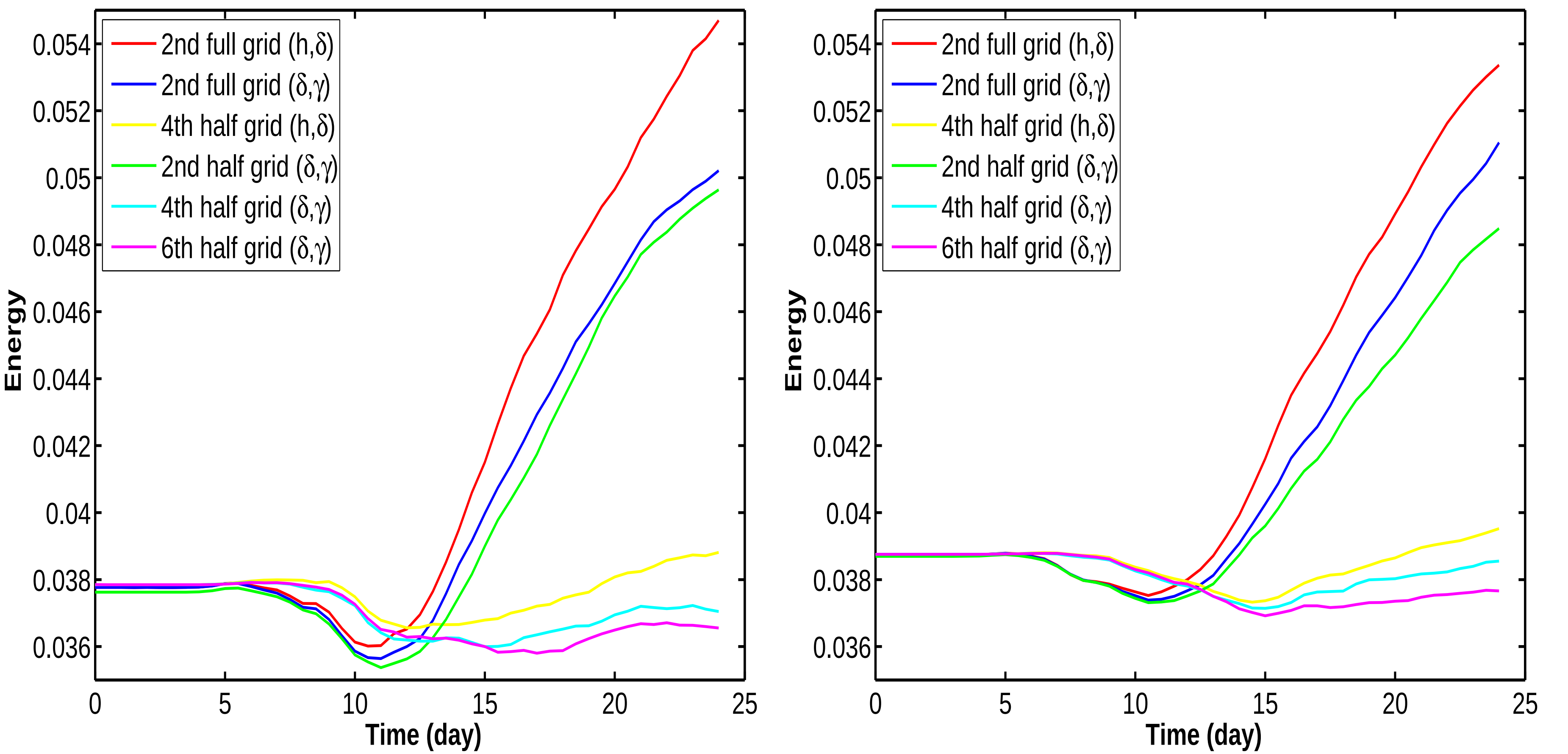
which goes unstable: 

Time evolution of potential vorticity (Q)



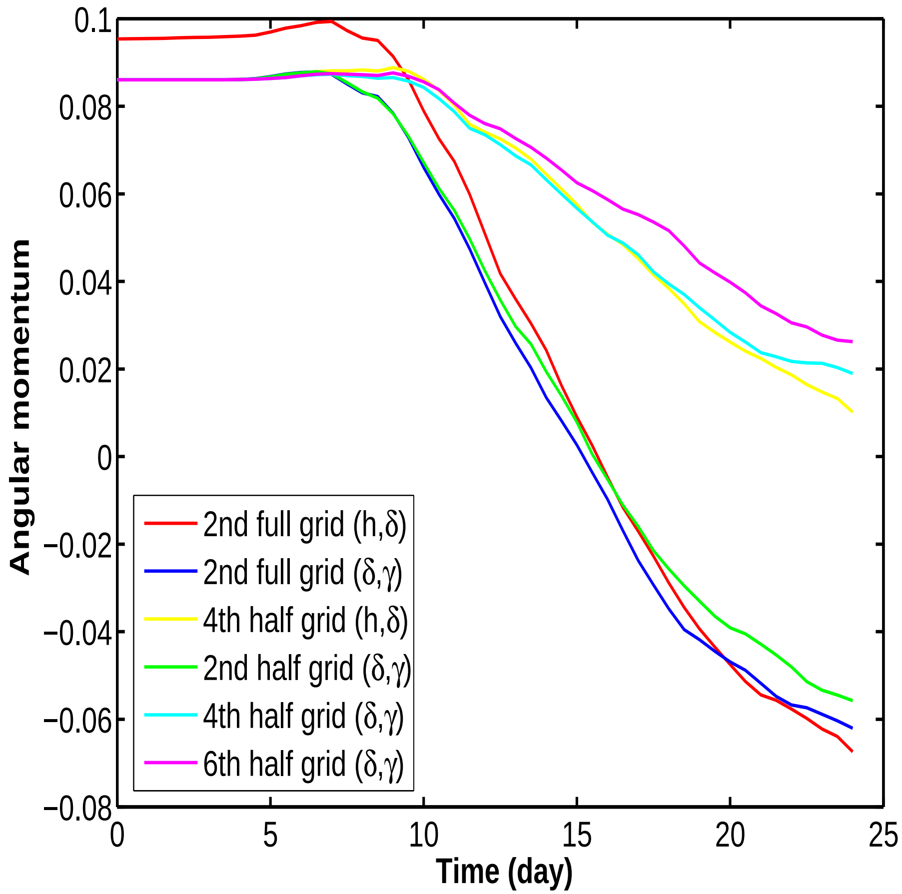
Energy conservation: $E = \langle \frac{1}{2}[(1+\tilde{h})(u^2+v^2) + c^2\tilde{h}^2] \rangle$

$(n_{\mathrm{g}} = 128)$ $(n_{\mathrm{g}} = 256)$

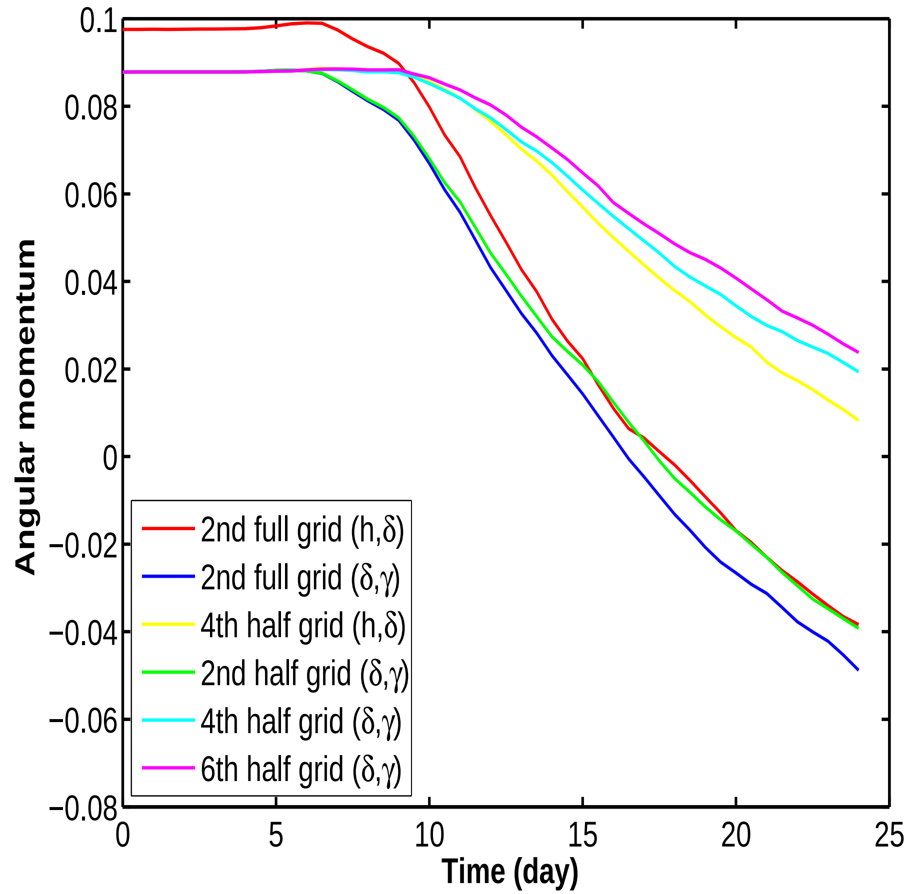


Angular momentum conservation: $J = \langle (1+\tilde{h})u + \Omega_{\mathrm{E}} \cos(\phi)\tilde{h} \rangle$

$(n_{\mathrm{g}} = 128)$



$(n_{\mathrm{g}} = 256)$



Imbalance

◇ Quantitative time evolution of imbalance as measured by *deviation* from Bolin–Charney balance:

$$\Xi = 0 \quad , \quad \frac{\partial \Xi}{\partial t} = 0$$

where

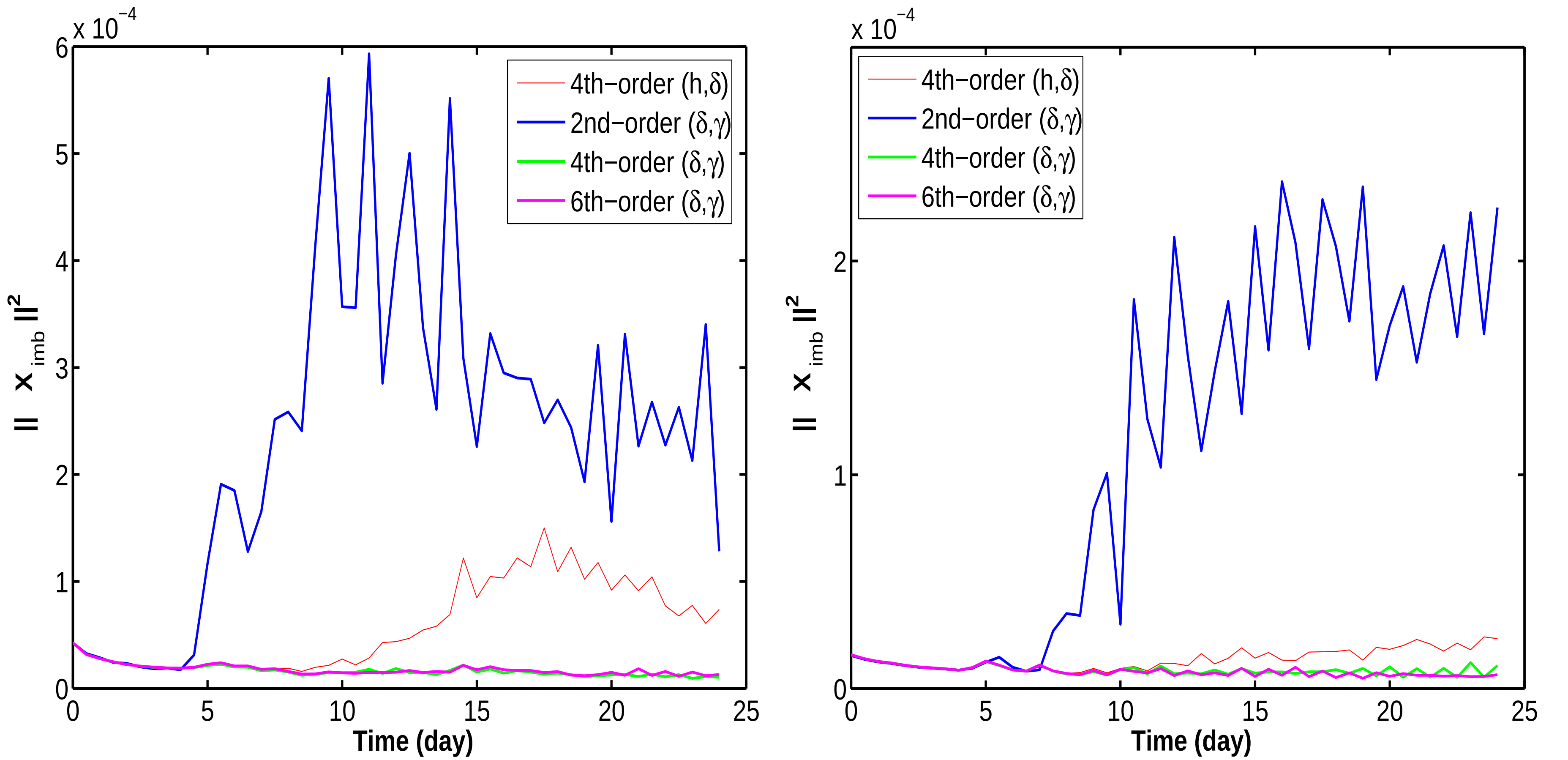
$$\Xi = (f\zeta - c^2\nabla^2\tilde{h} - \beta u_\psi) - 2\left[\frac{\partial u_\psi}{\partial \phi}\left(\frac{\partial u_\psi}{\partial \phi} + \zeta\right) + \left(\frac{\partial v_\psi}{\partial \phi}\right)^2\right] - (u_\psi^2 + v_\psi^2)$$

◇ Define the L_2 norm:

$$||\mathbf{X}|| = \left(\tfrac{1}{2}H\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}[c^2\tilde{h}^2 + (u^2 + v^2)]\,\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\lambda\right)^{1/2},$$

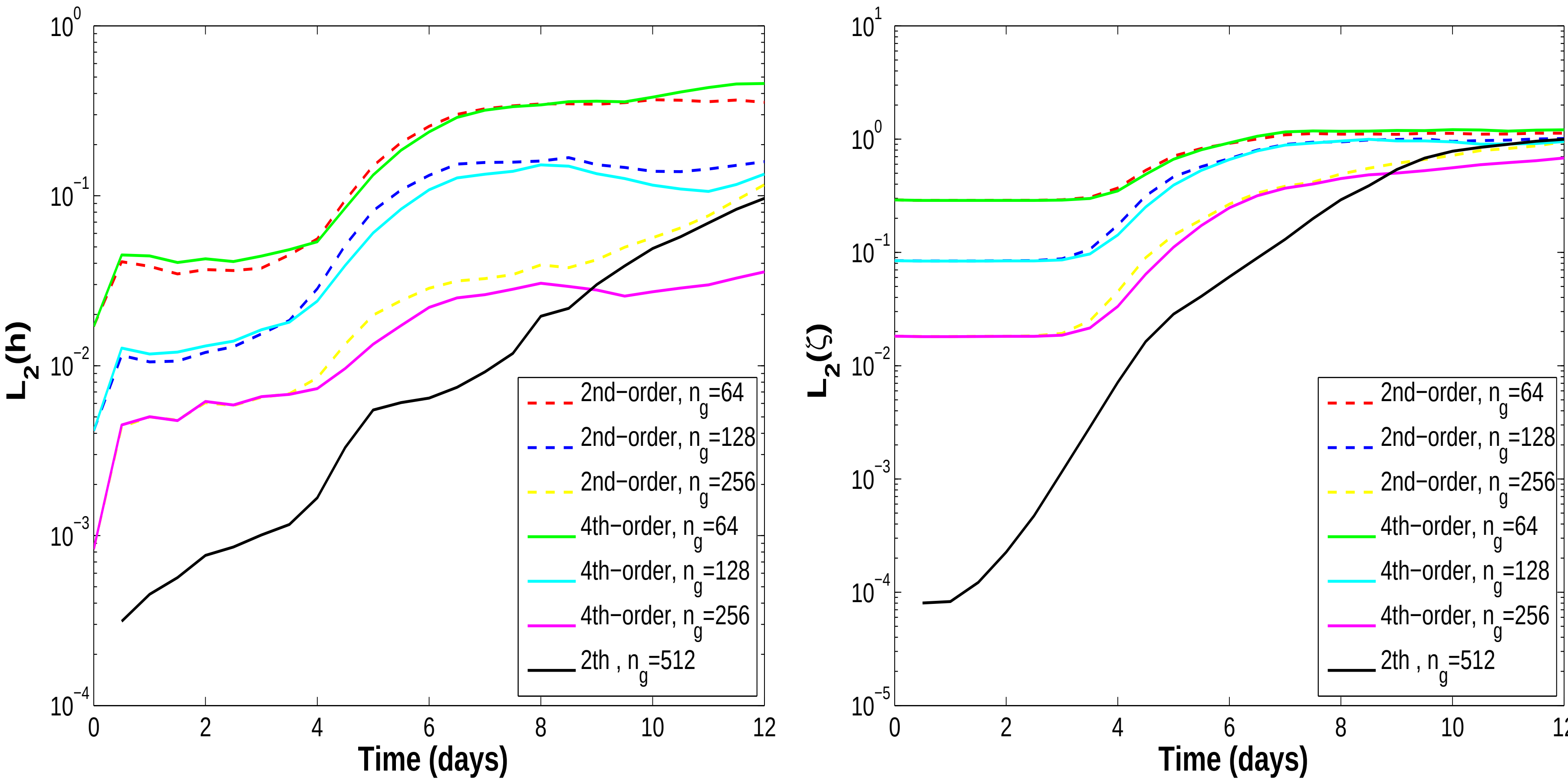
and measure imbalance by $||\mathbf{X}_{\mathrm{imb}}|| = ||\mathbf{X} - \mathbf{X}_{\mathrm{b}}||$, where $||\mathbf{X}||$, $||\mathbf{X}_{\mathrm{b}}||$, and $||\mathbf{X}_{\mathrm{imb}}||$ are the norms for the actual, balanced, and imbalanced state vectors.

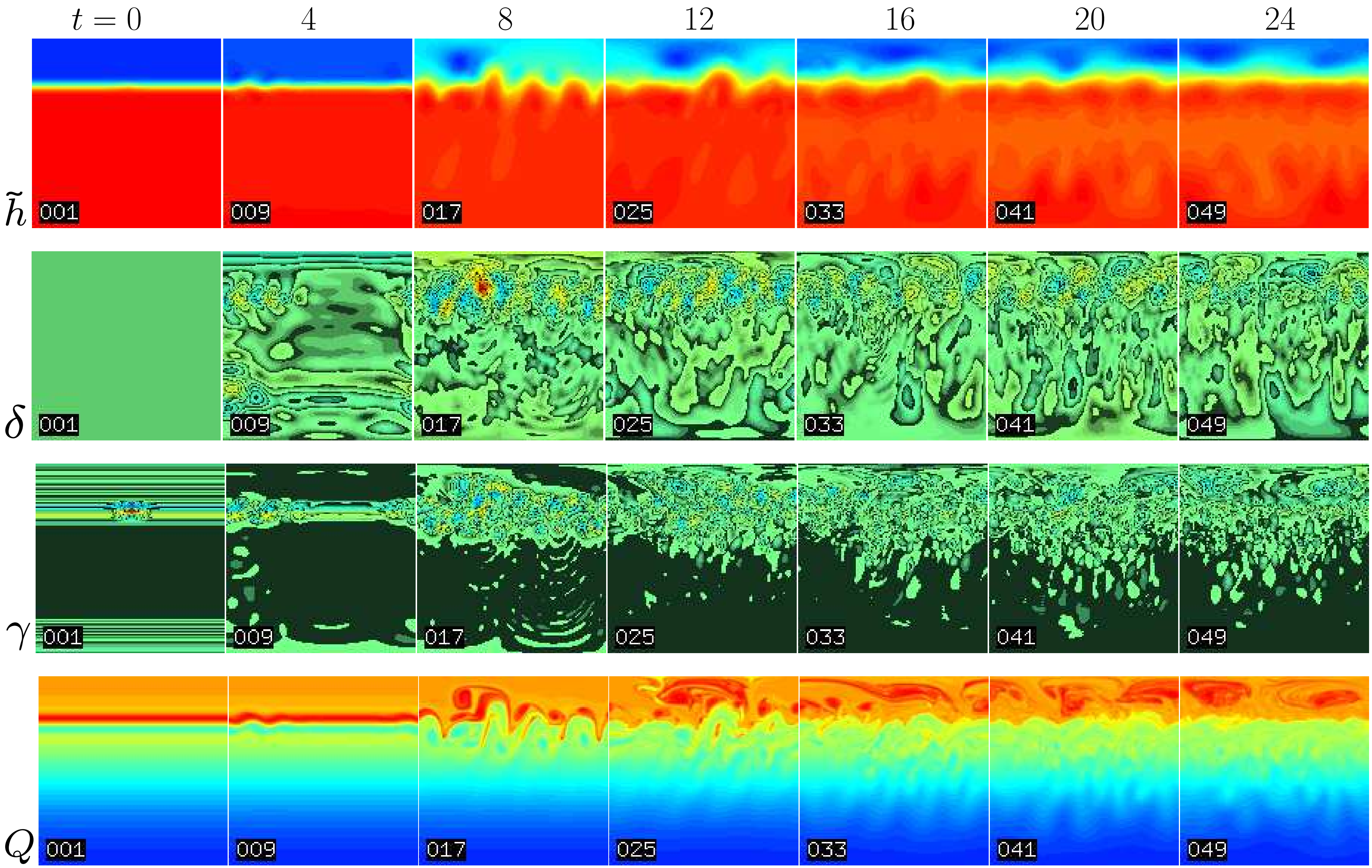
$(n_g = 128)$ $(n_g = 256)$


x 10^-4
4th−order (h,δ)
2nd−order (δ,γ)
4th−order (δ,γ)
6th−order (δ,γ)
|| x_imb ||^2
Time (day)
0
1
2
3
4
5
6
0
5
10
15
20
25
x 10^-4
4th−order (h,δ)
2nd−order (δ,γ)
4th−order (δ,γ)
6th−order (δ,γ)
|| x_imb ||^2
Time (day)

Convergence with resolution: measuring error by $L_2(\xi) = \frac{\left\{\langle|\xi(\lambda,\phi)-\xi_{\rm ref}(\lambda,\phi)|^2\rangle\right\}^{1/2}}{\left\{\langle|\xi_{\rm ref}(\lambda,\phi)|^2\rangle\right\}^{1/2}}$

(depth field) (vorticity field)

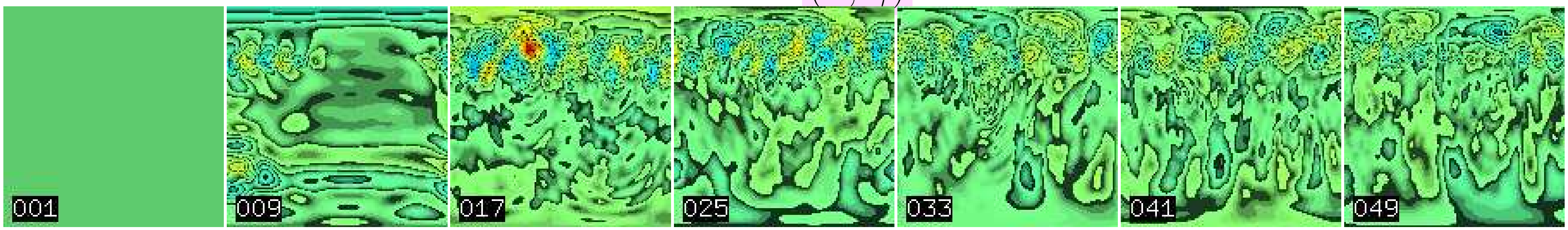




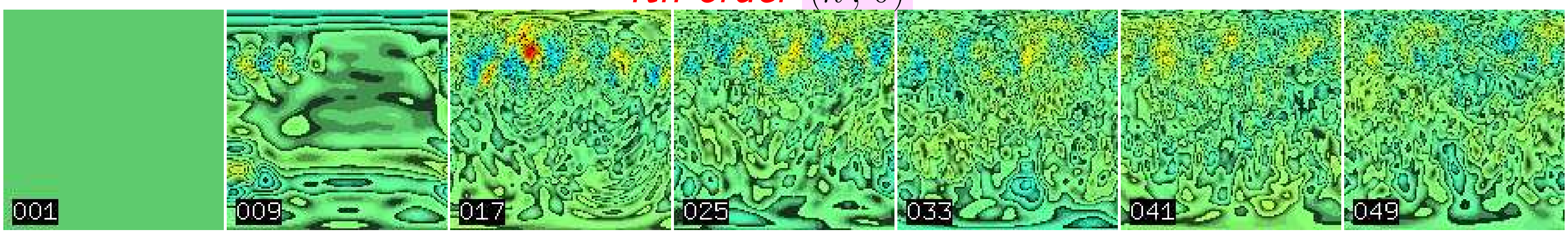
$t = 0$
4
8
12
16
20
24
$\tilde{h}$
δ
γ
Q
001
009
017
025
033
041
049

Divergence field

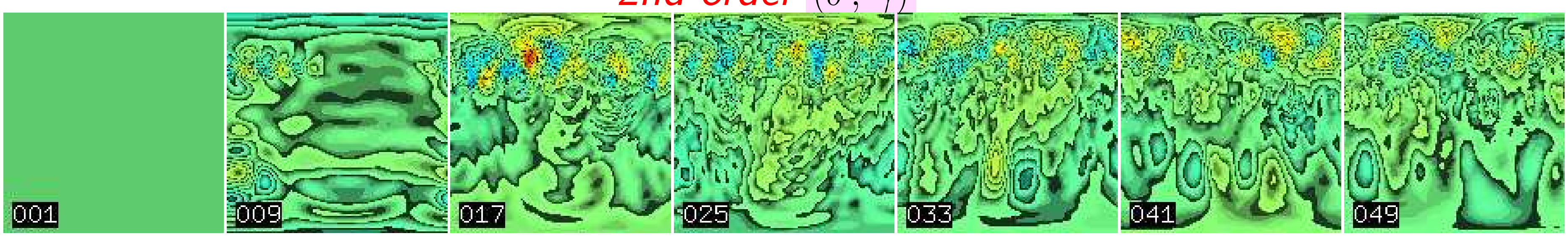
4th-order $(\delta\,,\,\gamma)$



4th-order $(h\,,\,\delta)$



2nd-order $(\delta\,,\,\gamma)$



Concluding Remarks

• Highly accurate modelling of *both* the PV-controlled balanced and the imbalanced parts of the flow by the contour-advective semi-Lagrangian (CASL) algorithm with prognostic variables: $(Q\,,\,\delta\,,\,\gamma)$

• Wave–vortex decomposition as a powerful tool to assess the accuracy of numerical algorithms

• The 4^{th}-order compact finite differencing in ϕ together with the pseudo-spectral treatment of derivatives in $\lambda \Rightarrow$ sufficient accuracy for the Eulerian part of the SW algorithm

• The (CASL) algorithm with prognostic variables $(Q\,,\,\delta\,,\,\gamma)$ has been extened to (i) inculde diabatic effects (non-conservative forcing) and (ii) multi-layer isentropic (isopycnal) model.