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Notes and Correspondence
On the momentum equation for the quasi-geostrophic model

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ABSTRACT: The momentum equation for the quasi-geostrophic (QG) model derived based on the conventional Rossby-number expansions does not uniquely determine the QG motion up to first order in the Rossby number. There are infinitely many ways of closing the equations. The momentum equation for QG derived by Holm and Zeitlin in 1998 based on a variational formulation for QG is compared with that for the conventional Rossby-number expansions. The underlying assumption in the construction of the variational formulation is geostrophic velocity for the particles. It is shown that the variational momentum equation corresponds to a particular way of closing the conventional momentum equation for QG. The numerical results for potential vorticity (PV) inversion on a circular vortex indicate a smaller range of applicability and loss of accuracy for the variational momentum equation for QG when compared with the QG one that sets the first-order linearized potential vorticity to zero. Copyright © 2009 Royal Meteorological Society

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1. Introduction

As a sequel to Mohebalhojeh (2002), this note is intended to compare the variational and standard momentum representations for the quasi-geostrophic (QG) model. Holm and Zeitlin (1998) present variational principles and a momentum equation for QG on the β plane, which will be referred to as the variational momentum equation for QG. There is also a standard, conventional momentum representation for QG based on small Rossby-number expansions. The conventional expansions suffer from an inherent indeterminacy beyond leading-order geostrophy (see Mohebalhojeh (2002) and the related discussions in Warn *et al.* (1995)). In the standard derivation of QG, the Rossby-number expansions are truncated at first order in the Rossby number. The result is a time-evolution equation for the QG potential vorticity (QGPV) and a set of diagnostic equations describing ageostrophic motion. Although QGPV is constructed using the geostrophic balance relation, as well as being advected by the leading-order geostrophic velocity, the first-order ageostrophic velocity is present in the standard momentum equation. As a matter of fact the diagnostic equations for the ageostrophic velocity are an indispensable part of QG theory. The presence of two velocity fields, one geostrophic and one ageostrophic, is a manifestation of 'velocity splitting' (McIntyre and Roulstone, 1996, 2002), a fundamental property shared by Hamiltonian and non-Hamiltonian balanced models with the exception of the hyperbalance potential-vorticity-based models (Mohebalhojeh and McIntyre, 2007).

The indeterminacy referred to above gives us the freedom to close the QG equations and thus use them in infinitely many ways for applications like the initialization of a primitive-equation model through QG, initialization of QG as a balanced model, and what is known as wave–vortex decomposition. In this note, the indeterminacy is exploited to show that the variational momentum equation derived by Holm and Zeitlin (1998) for QG corresponds to a particular way of closing the standard, conventional momentum equations for QG.

This note is organized as follows. In section 2 the standard derivation of QG is presented, highlighting the indeterminacy of the Rossby-number expansion. In section 3, the variational momentum equation for QG on an f plane is shown to correspond to a particular way of closing the conventional Rossby-number expansion expressed by a nonlinear relation between the first-order vorticity field and the zeroth-order height field. Section 4 demonstrates that the variational momentum equation entails both a loss of accuracy and a reduction in the range of applicability compared with a QG that sets the first-order linearized potential vorticity to zero. The final section makes remarks regarding the momentum equation for QG.

2. The quasi-geostrophic model

This model can be obtained from a standard small Rossby-number expansion of the shallow-water momentum and height equations (Pedlosky, 1987, section 3.12).

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The basic assumptions are that both the Rossby number, $\mathrm{Ro} = U/fL$, and the Froude number, $\mathrm{Fr} = U/\sqrt{gH}$, are small and of the same order, so that the Burger number, $\mathrm{Bu} = L_\mathrm{R}^2/L^2$, is of order unity. Here, U and L denote velocity- and length-scales, f is the Coriolis parameter, g the acceleration due to gravity, H the mean free-surface height and $L_\mathrm{R} = \sqrt{gH}/f$ the Rossby length, also called the Rossby deformation radius. To present the argument in the simplest case, the QG model here is presented for an f plane, i.e. f is taken to be constant. Expanding the velocity components u and v and the perturbation height h according to $(u, v, h) = (u, v, h)_0 + \mathrm{Ro}(u, v, h)_1 + \ldots$, to first order in Ro the equations read (Pedlosky, 1987, equation (3.12.19))

$$\frac{\mathrm{D}_0 u_0}{\mathrm{D}t} = -g\frac{\partial h_1}{\partial x} + f v_1, \quad (1a)$$

$$\frac{\mathrm{D}_0 v_0}{\mathrm{D}t} = -g\frac{\partial h_1}{\partial y} - f u_1, \quad (1b)$$

$$\frac{\mathrm{D}_0 h_0}{\mathrm{D}t} = -H\delta_1, \quad (1c)$$

where $\mathrm{D}_0/\mathrm{D}t = \partial/\partial t + u_0\partial/\partial x + v_0\partial/\partial y$ is the material derivative following $\mathbf{v}_0$, $fu_0 = -g\partial h_0/\partial y$, $fv_0 = g\partial h_0/\partial x$ and the zeroth-order divergence is $\delta_0 = 0$. The $\mathcal{O}(\mathrm{Ro})$ vorticity equation then becomes

$$\frac{\mathrm{D}_0 \zeta_0}{\mathrm{D}t} = -f\delta_1, \quad (2)$$

with the zeroth-order relative vorticity $\zeta_0 = (g/f)\nabla^2 h_0$. When combined with (1c), (2) leads to the conservation of QGPV denoted by q_g,

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\left(\zeta_0 - \frac{f}{H}h_0\right) \equiv \frac{\mathrm{D}_0}{\mathrm{D}t}(q_\mathrm{g}) = 0, \quad (3)$$

as well as a diagnostic equation for the first-order divergence δ_1,

$$\mathcal{H}\delta_1 = f\nabla\cdot(\mathbf{v}_0\zeta_0) - g\nabla^2\nabla\cdot(\mathbf{v}_0 h_0), \quad (4)$$

where $\mathcal{H} = gH\nabla^2 - f^2$ is the modified Helmholtz operator. The second term on the right-hand side of (4) is zero for the f-plane shallow-water QG model, but non-zero for multi-layer stratified models. The divergence equation consistent with (1a) and (1b) can be shown to be (see also Leith, 1980, equation (7.5))

$$(f\zeta_1 - g\nabla^2 h_1) \equiv \gamma_1 = -2J(u_0, v_0). \quad (5)$$

The QG balanced model, then, consists of the prognostic equation (3) for QGPV together with two diagnostic equations for δ_1 and γ_1. The curious property of the model is that δ_1 and γ_1, and thus the ageostrophic velocity fields, do not take part in the advection of QGPV. For this reason, the circulation associated with the ageostrophic velocity field is sometimes called the secondary circulation. However, the ageostrophic velocity is an indispensable part of QG. Without δ_1, the model reduces to the advection of geostrophic vorticity with geostrophic velocity. Notice that although (2) and (4) comprise a closed set of equations, the QG momentum equation remains unsolved in the sense that h_1 or ζ_1 cannot be determined unambiguously. To remove the ambiguity at this level of expansion, the equations are usually closed by some plausible assumption. The two most common assumptions have been to set either h_1 or ζ_1 to zero. Charney (1955) and Phillips (1960) have considered $h_1 = 0$ and $\zeta_1 = 0$, respectively, in the context of initialization of the primitive equations. Also, the QG momentum equation used by Hoskins *et al.* (1978) corresponds to $h_1 = 0$. The other choice coming from normal-mode considerations (Mohebalhojeh and Dritschel, 2001) is to set the first-order linearized potential vorticity to zero, that is $q_{\ell,1} \equiv \zeta_1 - (f/H)h_1 = 0$. Among the choices $h_1 = 0$, $\zeta_1 = 0$ and $q_{\ell,1} = 0$ to close the equations, the choice $q_{\ell,1} = 0$ produces the most accurate results (Mohebalhojeh, 2002) and will be used for comparison in section 4. This property carries over to higher order expansion in the Rossby number.† It is interesting to note that for the three-dimensional non-hydrostatic Boussinesq equations, McKiver and Dritschel (2008) obtain similarly excellent results from closing the equations by setting the first-order linearized potential vorticity to zero. Generally, one can consider an infinite set of functional relations between h_1 and ζ_1 such that the solutions of (5) satisfy the asymptotic requirement: $h_1/h_0 = \mathcal{O}(\mathrm{Ro})$ and $\zeta_1/\zeta_0 = \mathcal{O}(\mathrm{Ro})$. In any case, as far as the QG flow evolution is concerned, all those momentum-equation representations of QG consistent with (2), (4) and (5) are equivalent. However, for initialization and wave–vortex decomposition they may lead to different results.

3. The variational momentum equation for QG

Stating variational principles for QG in both Lagrangian and Eulerian representations, Holm and Zeitlin (1998) arrive at a momentum equation (see their equation (3.6)) that can be written in dimensional form for the f plane as

$$\frac{\partial}{\partial t}\mathcal{M}\mathbf{v} - \mathbf{v}\times\nabla\times(\mathcal{M}\mathbf{v} + \mathbf{R}) = -\nabla\left(gh + \frac{1}{2}\mathbf{v}\cdot\mathcal{M}\mathbf{v}\right), \quad (6)$$

where $\mathcal{M} = \left[1 - (f^2/gH)\nabla^{-2}\right] \equiv (1/gH)\nabla^{-2}\mathcal{H}$ and $\mathbf{R} = \hat{\mathbf{y}} f x$ such that the Coriolis parameter becomes $f = \hat{\mathbf{z}}\cdot\nabla\times\mathbf{R}$. Here $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the y and z directions, respectively. In this variational momentum equation, $\mathbf{v}$ is non-divergent ($\nabla\cdot\mathbf{v} = 0$) and thus $\mathbf{v} = \mathbf{v}_0$. Noting that $q_\mathrm{g} = \hat{\mathbf{z}}\cdot\nabla\times\mathcal{M}\mathbf{v}_0$, equation (6) can

†For the second-order expansion presented in Mohebalhojeh (2002), note that the second term of the right-hand side of equation (7c) in that paper should have g instead of f.

be rewritten in component form as

$$\frac{\partial}{\partial t}\mathcal{M}u_0 - v_0(f + q_g) = -\frac{\partial}{\partial x}(gh + \frac{1}{2}\mathbf{v}_0 \cdot \mathcal{M}\mathbf{v}_0), \quad (7a)$$

$$\frac{\partial}{\partial t}\mathcal{M}v_0 + u_0(f + q_g) = -\frac{\partial}{\partial y}(gh + \frac{1}{2}\mathbf{v}_0 \cdot \mathcal{M}\mathbf{v}_0), \quad (7b)$$

where $q_g \equiv q_{\ell,0} \equiv (\zeta_0 - \frac{f}{H}h_0)$. We recast (7) in the form

$$\frac{\partial}{\partial t}u_0 + \mathbf{v}_0 \cdot \nabla u_0 = F_v^1, \quad (8a)$$

$$\frac{\partial}{\partial t}v_0 + \mathbf{v}_0 \cdot \nabla v_0 = F_v^2, \quad (8b)$$

where $\mathbf{F}_v = (F_v^1, F_v^2)$ is used to denote the force field in the variational momentum equation. The variational force field can be recovered by comparing (7) and (8) with the result

$$\mathbf{F}_v = \mathcal{M}^{-1}[-\mathbf{v}_0 \cdot \nabla\mathbf{v}_0 + \frac{f}{H}\hat{\mathbf{z}} \times \mathbf{v}_0 h_0 - \nabla(gh_1 - \frac{f^2}{2gH}\mathbf{v}_0 \cdot \nabla^{-2}\mathbf{v}_0)] + \mathbf{v}_0 \cdot \nabla\mathbf{v}_0 \ . \quad (9)$$

We want to find the condition that makes $\mathbf{F}_v$ exactly the same as the force field given by (1), denoted by $\mathbf{F}_s$, i.e.

$$\mathbf{F}_s = -g\nabla h_1 - f\hat{\mathbf{z}} \times \mathbf{v}_1 \ \cdot \quad (10)$$

To this end, it is sufficient to check the curl and divergence of the two force fields. We have

$$\hat{\mathbf{z}} \cdot \nabla \times \mathbf{F}_s = -f\delta_1, \quad (11a)$$

$$\nabla \cdot \mathbf{F}_s = f\zeta_1 - g\nabla^2 h_1 \quad (11b)$$

for the standard formulation, and

$$\hat{\mathbf{z}} \cdot \nabla \times \mathbf{F}_v = -\mathcal{M}^{-1}\nabla \cdot (\mathbf{v}_0 q_{\ell,0}) + \nabla \cdot (\mathbf{v}_0\zeta_0), \quad (12a)$$

$$\nabla \cdot \mathbf{F}_v = \nabla \cdot (\mathbf{v}_0 \cdot \nabla\mathbf{v}_0) - \mathcal{M}^{-1}\nabla \cdot \left[\mathbf{v}_0 \cdot \nabla\mathbf{v}_0 - \frac{f}{H}\hat{\mathbf{z}} \times \mathbf{v}_0 h_0 + \nabla(gh_1 - \frac{f^2}{2gH}\mathbf{v}_0 \cdot \nabla^{-2}\mathbf{v}_0)\right] \quad (12b)$$

for the variational formulation.

Equating the two expressions (11a) and (12a), after some straightforward manipulation we obtain

$$f\mathcal{M}\delta_1 = (1 - \mathcal{M})[\nabla \cdot (\mathbf{v}_0\zeta_0)] - \frac{f}{H}\nabla \cdot (\mathbf{v}_0 h_0) \ .$$

Substituting $\mathcal{M} = (1/gH)\nabla^{-2}\mathcal{H}$ for $\mathcal{M}$ and acting on both sides by $(gH/f)\nabla^2$ gives the QG divergence equation (4). Therefore, the variational momentum equation is consistent with the standard momentum equation for QG as regards the resulting diagnostic equation for the divergence field.

Equating the two expressions (11b) and (12b), noting that $\nabla \cdot (\mathbf{v}_0 \cdot \nabla\mathbf{v}_0) = -2J(u_0, v_0)$, it is straightforward to derive

$$f\zeta_1 = -\frac{g}{2H}\nabla^2(h_0^2 - \nabla h_0 \cdot \nabla\nabla^{-2}h_0) \quad (13)$$

as the condition for the consistency of the divergence of the variational and standard force fields. Incidentally, (13) is the condition that the evolution of (u_0, v_0) by (7) remains non-divergent. Given the quadratic nature of the right-hand side, (13) seems to respect the QG scaling. Equation (13) serves as the particular way of closing the conventional momentum equation for QG that makes it equivalent to the variational momentum equation.

4. A numerical examination

In a doubly periodic $2\pi \times 2\pi$ domain, a torus, a circular vortex at the centre of the domain, is constructed. The vortex is characterized by a uniform patch of Rossby–Ertel potential vorticity $Q = (f + \zeta)/(H + h)$. For the periodic domain, given that the vortex is located far away from the boundaries, to a very good approximation the vortex is in a steady state in cyclostrophic balance. The PV is taken to be uniform both inside and outside the patch. The jump in PV between the outside and inside of the vortex denoted by ΔQ takes the values $\{0.01, 0.02, 0.03, \ldots, 1.9, 2.0\}f/H$, spanning Rossby numbers that are small to order of one. For each PV jump, the Rossby–Ertel PV is inverted by the highly accurate third-order $\delta\delta$ balance (Mohebalhojeh and Dritschel, 2001) to reconstruct the mass and velocity distribution of the the nearly steady vortex in cyclostrophic balance. The third-order $\delta\delta$ PV inversion solution obtained by setting the second and third time derivatives of the divergence to zero is taken as the reference solution. The large span for the PV jump is to assess the range of validity of the QG relations. Then PV is inverted by the diagnostic equations of QG and (i) setting $q_\ell = q_{\ell,0}$, (ii) using (13), which is the variational QG.

The results for the relative error against the PV jump shown in Figure 1 reveal two main properties of the variational QG, that is, a smaller range of applicability and noticeably lower accuracy when compared with the QG that sets $q_{\ell,1} = 0$. The iterative solution procedure designed for the variational QG exhibits weak convergence even for sufficiently small Ro, and for $H\Delta Q/f > 0.47$ it stops converging. The convergence problem of the variational QG is not an artefact of the iterative solution procedure, but is a direct consequence of the nonlinear terms present on the right-hand side of (13).

5. Concluding remarks

At the heart of the problem considered is the indeterminacy of defining a velocity field for the particles in QG dynamics. Although QGPV is advected by

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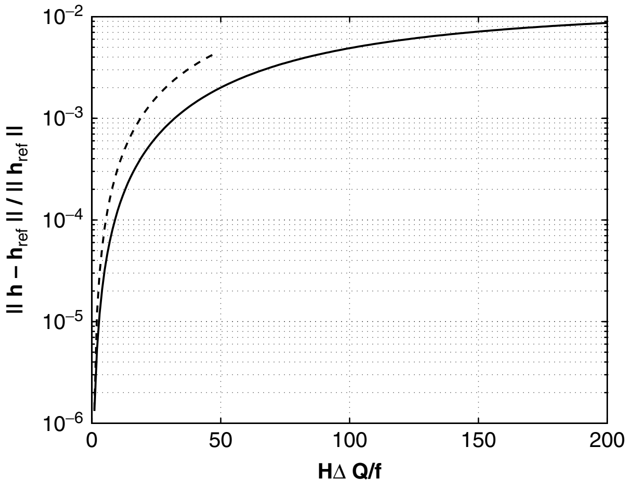

Figure 1. The relative errors of the variational QG (dashed) and the QG solution with $Q_{\ell,1} = 0$ (solid) for $H\Delta Q/f = [0.01, 2.0]$. The reference solution uses the third-order $\delta\delta$ balance.

the geostrophic velocity, unlike the case for an exact geostrophic state, the acceleration is not generally zero. The geostrophic velocity for the particles assumed in the Holm and Zeitlin (1998) variational formulation of QG is not the only $\mathcal{O}(\mathrm{Ro})$-consistent velocity for particles. As shown here, there are in principle an infinite number of particle velocities consistent with the QG momentum equation.

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