Super Compact Spatial Differencing for the Linear and Nonlinear Geophysical Fluid Dynamics Problems

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Abstract

The Super Compact Finite Difference Method (SCFDM) is applied to spatial differencing of some prototype linear and nonlinear geophysical fluid dynamics problems. For the frequency of linear inertia-gravity waves on different numerical grids (Arakawa's A-E and Randall's Z), the sixth-order SCFDM shows a substantial improvement on the conventional methods. For the Jacobians involved in vorticity advection by nondivergent flow and in Bolin–Charney balance equation, as nonlinear problems, it is found that the sixth-order SCFDM provides a noticeably more accurate representation of the wavenumber distribution of the Jacobians, when compared with the conventional methods. In addition, computation of a normalized global error at different horizontal resolutions in longitude and latitude directions for the Rossby–Haurwitz wave on a sphere shows that sixth-order SCFDM can markedly improve on the fourth-order compact.

1 Introduction

The compact finite difference schemes, introduced as far back as the 1930s, have been found as simple, yet powerful ways of reaching the objectives of high accuracy and low computational cost [14, 12]. A recent development in compact schemes has been the introduction of a general procedure to generate highly accurate schemes of arbitrary order with minimal stencil size by Fu Dexun and Ma Yanwen [4], who called it Super Compact Finite Difference Method (SCFDM). The derivation and the application of the method in uniform grid have been presented in [16, 5]. In addition, some aspects of the scheme in nonuniform grid have been investigated by Ghader [9] and Esfahanian *et al.* [7].

This paper is devoted to the assessment of the accuracy of SCFDM in the atmosphere–ocean dynamics context. We apply the sixth order SCFDM to spatial differencing of some prototype linear and nonlinear geophysical fluid dynamics problems.

2 Super compact method

The super compact comprises of a basic equation and a set of auxiliary equations. The basic equation is obtained by truncating the Taylor series which relates a function (ϕ) to its derivatives. To arrive at a closed system of equations, a set of $(n-1)$ additional independent equations called auxiliary equations, is needed. The SCFDM relations can be written in a vector form:

$$-\frac{1}{2}\mathbf{L}(\frac{1}{\sigma_j})\mathbf{F}_{j-1}+(\mathbf{A}+\mathbf{L})(\sigma_j)\mathbf{F}_j-\frac{1}{2}\mathbf{L}(\sigma_{j+1})\mathbf{F}_{j+1}=\frac{1}{2}(\alpha\Delta_x^+ +\beta\Delta_x^-)\phi_j\mathbf{E} \quad (1)$$

in which α and β are free parameters, $\Delta_x^+\phi_j=\phi_{j+1}-\phi_j$, $\Delta_x^-\phi_j=\phi_j-\phi_{j-1}$ and the definition of the vectors and matrices are given in [4, 7]. At the boundaries, forward and backward relations are used [16, 9].

The derivation of a more convenient form of the SCFDM relations is also possible [7]. The alternative form of the sixth-order approximations of the first and second derivatives are obtained as

$$\phi_j^{<1>}=\left(\frac{10(12+\Delta_x^2)\Delta_x^\circ}{120+30\Delta_x^2+\Delta_x^4}\right)\phi_j, \qquad \phi_j^{<2>}=\left(\frac{30(12+\Delta_x^2)\Delta_x^2}{360+60\Delta_x^2+\Delta_x^4}\right)\phi_j \quad (2)$$

where $\Delta_x^\circ=(\Delta_x^+ +\Delta_x^-)/2$, $\Delta_x^2=\Delta_x^+\Delta_x^-=\Delta_x^-\Delta_x^+$. The sixth-order approximation of the first derivative for the cell-centered can be found as

$$\phi_j^{<1>}=\left(\frac{320(12+\Delta_x^2)}{1920+240\Delta_x^2+\Delta_x^4}\right)(\phi_{j+\frac{1}{2}}-\phi_{j-\frac{1}{2}})/2 \quad (3)$$

3 Spatial differencing of the linear problems

For the spatial differencing of the linear problems using SCFDM, the linearized shallow water equations on an f-plane are considered. The shallow water equations will be discretized on the well known Arakawa A-E grids [1], and on the Randall's Z grid [13].

By assuming wave solutions, for the continuous linearized shallow water equations [1, 13] the dispersion relation $(\omega/f_\circ)^2=1+\lambda^2(\kappa^2+\ell^2)$ is obtained. In this equation ω is the frequency, $\lambda=\sqrt{gH}/f_\circ$ is the Rossby radius of deformation, and κ and ℓ are the wavenumbers in the x and y directions, respectively.

To represent the discrete dispersion relationships for the inertia-gravity waves on Arakawa A-E grids, the generic expressions of the discrete dispersion relationships derived by Blayo [2] are used. This idea, here, can be extended in a similar way to obtain the discrete dispersion relation $(\omega/f_\circ)^2 = 1 - \lambda^2 [T_2(\kappa) + T_2(\ell)]$ for the Z grid. We note that the transfer function, T of a scheme S is defined by $S(e^{i\kappa x}) = T(\kappa)e^{i\kappa x}$.

The results of the discrete relations of the different methods on the numerical A-E and Z grids are presented for a resolved ($\lambda/d = 2$) and for an under-resolved ($\lambda/d = 1/2$) cases. Figure 1 show the 10% relative error contour for the sixth-order SCFDM. The relative error is defined as $|\omega_\mathrm{e} - \omega_\mathrm{n}|/|\omega_\mathrm{e}|$, where ω_e is the exact frequency and ω_n is the grid frequency which is computed for different methods on A-E and Z grids. By comparing these results

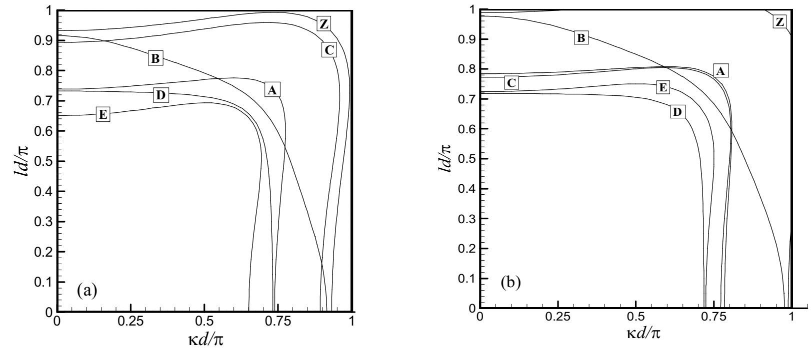

Fig. 1. Contour lines of 10% error in frequency for the dispersion relationship of the inertia-gravity wave on A-E and Z grids obtained from the sixth-order super compact method, (a) $\lambda/d = 2$ and (b) $\lambda/d = 1/2$.

with those obtained from the second-order centered [6] and the fourth-order compact [2] methods, it can be seen that the sixth-order SCFDM leads to a very clear increase in the accuracy of the frequency of inertia-gravity waves.

4 Spatial differencing of nonlinear problems

In this part, we study the two Jacobians appearing in the vorticity and divergence equations as the dominant nonlinear terms. The Jacobians are $J(\psi, \zeta)$ and $J(u_\psi, v_\psi)$, where ψ is the streamfunction, $u_\psi = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$, and $J(X, Y) = (\partial X/\partial x)(\partial Y/\partial y) - (\partial X/\partial y)(\partial Y/\partial x)$ for any two arbitrary quantities X and Y. It is assumed that the streamfunction ψ, and vorticity ζ, are on an unstaggered grid. The different, though identical in continuous case, forms of the Jacobian operator for vorticity advection are given in [11, 10]. It should be emphasized that the Jacobian operators are different when discretized by numerical methods. Following Chang and Shirer [3], for a doubly periodic domain, the continuous ψ is expanded in a complex Fourier series to find the analytical and discrete forms of two Jacobians.

In this paper, only the results for the amplitude errors of $J_\mathrm{A}(\psi, \zeta)$ and $J_1(u_\psi, v_\psi)$ are presented. We assume that ψ is composed of two components having different wanenumbers m_1 and m_2 in the x-direction and wavenumber n in the y-direction [3] i.e., $\psi = A_{m_1,n}\exp[\mathrm{i}(m_1x+ny)] + A_{m_2,n}\exp[\mathrm{i}(m_2x+ny)]$.

To evaluate the exact and approximate Jacobians, it is appropriate to make a physically meaningful model for the amplitude $A_{m,n}$. To recover the amplitude for streamfunction, we use quasigeostrophic relations in the potentio-vortical energy E in spectral space given in [17], and obtain $A_{m,n} = \left(2E/(H(m^2+n^2+1/\lambda^2))\right)^{1/2}$.

Figure 2 presents the 5% relative errors for $J_{\mathrm{A}}(\psi,\zeta)$ and $J_1(u_\psi,v_\psi)$ and the three numerical schemes [amplitude errors for the wave $(m_1+m_2,2n)$]. Notice that the error increases from small to large values of m_1 and m_2 almost isotropically, except for a reduction at large values of the wavenumbers at the upper right-hand corner for $J_1(u_\psi,v_\psi)$. It is clear that the fourth-order compact method gives a significant improvement on the second-order centered. Further, the sixth-order SCFDM enjoys a noticeably larger region with relative error less than 5% compared with the fourth-order compact. It is expected that the use of sixth-order SCFDM will further improve the representation of wave-wave interactions.

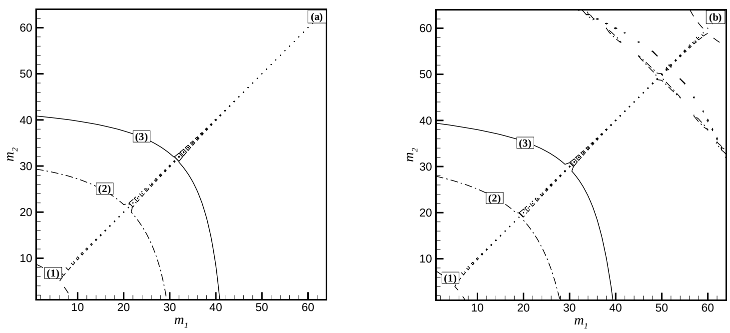

Fig. 2. The contours represent the 5% relative error for (a) $J_{\mathrm{A}}(\psi,\zeta)$ and (b) $J_1(u_\psi,v_\psi)$. The dashed, dashed-dotted, and solid lines are for, respectively, (1) the second-order centered, (2) the fourth-order compact, and (3) the sixth-order SCFDM.

5 Spherical geometry

In this section, the problem of reconstruction of the streamfunction field from the vorticity on the sphere is considered. A well-known analytical solution is used to compare the results of the three schemes. For the analytical case the Rossby–Haurwitz wave, which is one of the standard tests proposed by Williamson *et al.* [15] is used. To find the streamfunction from the known vorticity field a Poisson equation on the sphere must be solved. The Poisson equation is solved on a doubly periodic longitude-latitude grid system. The longitude-latitude grid system of Fornberg [8] and Nihei and Ishii [12] is used for the computation.

The second-order centered, fourth-order compact and sixth-order SCFDM schemes are used to discretize the spatial derivatives to find the numerical streamfunction filed of the Rossby–Haurwitz wave. Spatial discretization of the derivatives are performed in both longitude and latitude directions which leads to solve a cyclic block tridiagonal system in each direction for the fourth-order compact and the sixth-order SCFDM and a cyclic tridiagonal system for the second order centered. These solutions are compared with the analytical streamfunction of the Rossby–Haurwitz wave to find the error of each scheme. Figure 3 shows the normalized global error l_2 [15] for $\partial\psi/\partial\varphi$ at different horizontal resolutions in longitude and latitude directions, calculated for the second-order centered, fourth-order compact, and sixth-order SCFDM. It can be seen that the sixth-order SCFDM gives a marked improvement in accuracy on the fourth-order compact, almost the same as the improvement shown by the fourth-order compact on the scend-order centered. It is worth mentioning that the convergence rates shown by the three Poisson solutions are consistent with their order of accuracy.

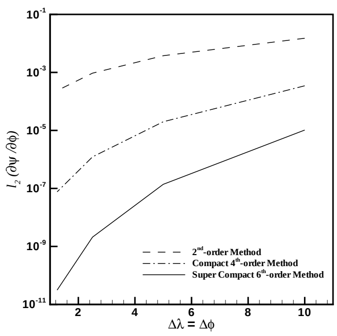

Fig. 3. Normalized l_2 error of $\partial\psi/\partial\varphi$ field as a function of resolution obtained using the second-order centered, fourth-order compact, and sixth-order SCFDM for the Rossby–Haurwitz wave. A uniform resolution, shown in degrees in the figure, is used in both λ and φ directions.

6 Concluding remarks

For the linear propagation of inertia-gravity waves on the f-plane, the sixth-order SCFDM shows a significant improvement on the conventional second-order centered and the fourth-order compact. As a remark, we expect the same for the linear propagation of Rossby waves on the β-plane and on the sphere. For the nonlinear problems represented by the Jacobians involved in vorticity advection by nondivergent flow and in Bolin–Charney balance equation, the improvement by the sixth-order SCFDM on the fourth-order compact in wavenumber distribution of the Jacobians is noticeable, amounting to nearly a 25% reduction in the global error measures calculated. For the reconstruction of the streamfunction of the vorticity field of the Rossby–Haurwitz wave, the sixth-order SCFDM improves markedly on the fourth-order compact.

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